Deforming Lagrangian submanifolds by functions of their angle

Von der Fakultät für Mathematik und Physik

der Gottfried Wilhelm Leibniz Universität Hannover

zur Erlangung des Grades

Docktor der Naturwissenschaften

Dr. rer. nat.

genehmigte Dissertation

von

Diplome d'Etudes Approfondies (D.E.A.) Wassih Gilbert Laval Roger
Adémola MARCOS geboren am 04.02.1980 in Porto Novo, Benin

Referent: Prof Dr. Knut Smoczyk

Korreferent: PD Dr. Lutz Habermann

Tag der Promotion: 26. Oktober 2009

Wassih Marcos

Die Deformation von Lagrange Untermannigfaltigkeiten entlang ihres Lagrange-Winkels

November 9, 2009

Leibniz Universität Hannover

To my late mother

Rachidath Bello Thiamiyou

and my father

Ganiyou Marcos.

Acknowledgment. This research project would not have been possible without the support of many people. I wish to express my gratitude to my supervisor, Prof. Dr. Knut Smoczyk, who was abundantly helpful and offered invaluable assistance, support and guidance. Special thanks also to all my friends and colleagues, especially to the members of the *Institut für Differentialgeometrie*; Dr. Matthias Bergner, Mykhaylo Chursin, PD Dr. Lutz Habermann, Dr. Lars Schäfer and Maren Stroot for sharing the literature and invaluable assistance. I would also like to convey thanks to Ms. Melanie Schunert for invaluable assistance with administartive matters and the Faculty for providing the financial means and facilities. I thank also the IMSP (Institut de Mathématique et de Sciences Physiques) in Benin and Dr. Léonard Todjihoundé for his advice and his support for the good evolution of my thesis.

Zusammenfassung

Ein Ziel in der Geometrie war und ist es, minimale Untermannigfaltigkeiten zu konstruieren. Ein Weg ist die Deformation einer gegebenen Untermannigfaltigkeit entlang ihres mittleren Krmmungsvektors. Dieser so genannte mittlere Krmmungsfluss ist zwar nicht der einzige mögliche Fluss, um minimale Untermannigfaltigkeiten zu erzeugen, aber der beste, da unter diesem Fluss das Volumen der Mannigfaltigkeit am schnellsten verringert wird. Auch unter anderen Flssen enstehen minimale Untermannigfaltigkeiten zum Beispiel unter Donaldsons Fluss. In der vorliegenden Dissertation untersuchen wir Lagrangesche Untermannigfaltigkeiten unter durch Funktionen des Langrange Winkels definierten Flssen und versuchen so minimale Untermannigfaltigkeiten zu erzeugen.
Stichwort: Lagrange Untermannigfaltigkeiten, Lagrange-Winkels, Maslov class.

Abstract

Geometers have been interested in constructing minimal submanifolds for a long time. One possible way is to deform a given submanifold by its mean curvature vector. The so called mean curvature flow is not the only possible flow to produce minimal submanifolds but the best one because it represents the most effective way to decrease the volume of a submanifold. There are many other possible ways to get minimal submanifolds, Donaldson's flow [48], for example. In this thesis, we also try to get minimal submanifold by deforming Lagrangian submanifolds by functions of their angle.
Keywords: Lagrangian submanifolds, Lagrangian angle, Maslov class.

Contents

1

Introduction

A smooth flow on a smooth manifold M is a smooth $\mathbb{R}$-action $\mu : \mathbb{R} \times M \to M$. By setting $\mu_t(x) := \mu(t, x)$, for each $t \in \mathbb{R}$, a flow μ defines a one-parameter group $(\mu_t)_{(t\in\mathbb{R})}$ of transformations of M. If the diffeomorphism $\mu_t : M \to M$ for each $t \in \mathbb{R}$ preserves some geometric structure given on M, then μ is called a geometric flow.

Deforming submanifolds via various geometric parabolic flows has been a powerful method in differential geometry. In [3], Brakke introduced the motion of a submanifold moving by its mean curvature in arbitrary codimension and constructed a generalized varifold solution for all time. This geometric flow is called mean curvature flow. There are many other geometric flows. For example, the harmonic map heat flow is the most famous geometric flow to deform the maps between Riemannian manifold. It is precisely the gradient flow of the energy functional of maps.

Geometers have been interested in constructing minimal submanifolds for a long time. One possible way to produce such submanifolds is to deform a given submanifold by its mean curvature vector. More precisely, one might evolve the submanifold by the negative gradient flow of the volume functional. This gives the well known mean curvature flow equation

$$\frac{dF}{dt}(x,t) = \overrightarrow{H}(x,t)\,, \tag{1.1}$$

where $F : M \times [0,T) \to N$, $T > 0$, is a smooth family of immersions of a given smooth manifold M into some Riemannian manifold (N, g) and $\overrightarrow{H}(x,t)$ denotes the mean curvature vector of M at $(x,t) \in M \times \{t\}$, $t \in [0,T)$.

(1.1) is a quasi-linear parabolic equation (see for example [13]) and the parabolic theory implies the mean curvature flow (1.1) has a smooth solution for short time if the initial submanifold M_0 has bounded curvature. More precisely, there exists $T > 0$ such that (1.1) has a smooth solution in the time interval $[0,T)$. If the second fundamental form A on $M_t := F(M,t)$ is uniformly bounded in t near T, then the solution can be extended smoothly to $[0, T+\epsilon)$ for some $\epsilon > 0$.

However, in general $\max_{M_t} |A|^2$ becomes unbounded as $t \to T$. In this case we say that the mean curvature flow blows up at T; moreover, to classify the singularities of the mean curvature flow of hypersurfaces, Huisken, according to the blow-up rate of $|A|$, introduced the following notion: Suppose M_t is a solution of the mean curvature flow and suppose that $T > \infty$ and

$$\limsup_{t \to T} |A|^2 = \infty.$$

Then, if there exists a positive constant C such that

$$\sup_{M_t} |A|^2 \leq \frac{C}{T-t},$$

we say that the mean curvature flow develops a Type-I singularity at T; otherwise the singularity will be called type-II.

In case $N = \mathbb{R}^{n+1}$, it has been shown by [27] that solutions of (1.1) forming a Type-I singularity can be homothetically rescaled so that any resulting limiting submanifold satisfies

$$H = -F^{\perp}. \tag{1.2}$$

The solutions of (1.2) are called self-similar solutions or more precisely self-shrinking solutions of the mean curvature flow. Abresch and Langer [1] have completely classified self-similar curves $\Gamma \subset \mathbb{R}^2$ and this result can be applied to curves $\Gamma \subset \mathbb{R}^n$ equally well. Huisken classified all self-shrinking hypersurfaces in $\mathbb{R}^{n+1}$ with $H > 0$ [28]. In [46], Smoczyk extended this result and proved that a self-shrinker in arbitrary codimension is a minimal submanifold of the sphere, if and only if $\overrightarrow{H} \neq 0$ and the principal normal is parallel.

(1.1) represents the most effective way to decrease the volume of a submanifold in the sense that it is the negative gradient flow of the volume functional. For the classical solution of the mean curvature flow, most work has been done for hypersurfaces, in particular in euclidean space.

As mentioned above, the motion of surfaces by their mean curvature has been studied by Brakke [3] from the viewpoint of geometric measure theory. Other authors investigated the corresponding nonparametric problem (see [10], [17], [51]). Huisken showed in ([26]) that if the initial hypersurface is compact and uniformly convex in a complete manifold with bounded geometry then it converges to a single point under the mean curvature flow in finite time and the normalized flow (volume is fixed) converges to a sphere of that volume in infinite (rescaled) time. Another classical result is due to Ecker and Huisken ([11]), where they study hypersurfaces in $\mathbb{R}^{n+1}$ that can be represented as entire graphs over a flat plane. Their result says that any polynomial growth rate for the height and the gradient of the initial hypersurface M_0 is preserved during the evolution and that in case of Lipschitz initial data with linear growth, the mean curvature flow has a smooth solution for all time.

However, as time evolves, the mean curvature flow may develop singularities which can be classified as type-I and type-II singularities according to the blow-up rate of the second fundamental form with respect to time t (cf. [25]). The existence, uniqueness, and regularity of a weak solution of the mean curvature flow (the so-called viscosity solution) were studied by Chen-Giga-Goto [6], Evans and Spruck [14], White [55], and others.

In general, the flow becomes much more complicated, if the codimension increases. This is partially caused by the fact that the normal bundle $T^{\perp}M$ of M in N might no longer be intrinsic, as is the case for hypersurfaces.

One interesting example of a nice mean curvature flow in higher codimension is the Lagrangian mean curvature flow. This is the mean curvature flow under the extra assumption that the initial submanifold is a Lagrangian submanifold in a Kähler-Einstein manifold. Lagrangian submanifolds are n-dimensional submanifolds L in $2n$-dimensional symplectic manifolds $(M, \bar{\omega})$ such that $\omega := \bar{\omega}_{TL} = 0$. There are important in physics and of course in pure and applied mathematics as well. E.g., if M is a Calabi-Yau 3-fold, then $H^{1,1}(M)$ and $H^{2,1}(M)$ can be recovered from associated Conformal Field Theories as eigenspaces of a certain operator. The only difference between the Conformal Field Theories representations for $H^{1,1}(M)$ and $H^{2,1}(M)$ is the sign of their eigenvalue under a $U(1)$-action. Since the sign is only a matter of convention, this led some physicists to conjecture that there should exist a Calabi-Yau 3-fold $\bar{M}$ with the same Conformal Field Theories but with different signs for the operators, so that $H^{1,1}(M) = H^{2,1}(\bar{M})$ and the mirror conjecture was born.

It was shown in [42] by Smoczyk (see also [53]) that the Lagrangian mean curvature flow preserves the Lagrangian condition in such cases. The flow becomes interesting for two main reasons. The first is, that the Lagrangian condition implies that the tangent and normal bundles of M are isometric, so that again (as in the hypersurface case) the normal bundle can be viewed intrinsically. The second reason is, that the Lagrangian condition means that the flow equation (1.1) can be (locally) integrated and the quasi-linear system (1.1) will generate a single fully nonlinear parabolic equation of Monge-Ampère type. In particular, in the regularity theory one gains one degree of differentiability.

In [42], Smoczyk considered Lagrangian submanifolds generated by symplectic maps and proved convergence to minimal Lagrangian maps under very natural and sharp conditions for the Lagrangian angle α. In [43] he gave some crucial condition based on terms of certain symmetric bilinear forms S in flat ambiant manifolds. He got then a longtime existence and convergence smoothly to a flat Lagrangian submanifold. In [45] Smoczyk and Wang proved that if the potential function of a Lagrangian torus in T^{2n} is convex, then the flow exists for all time and converges smoothly to a flat Lagrangian submanifold.

Based on methods from geometric measure theory, Neves proved in [35] that Lagrangian submanifolds with zero Maslov class do not develop type-I singularities in finite time. K. Groh, M. Schwarz, K. Smoczyk and K. Zehmisch in [20], used holomorphic disks to describe the formation of singularities in mean curvature flow of monotone Lagrangian submanifolds in $\mathbb{C}^n$. In this paper they showed that the flow preserves the monotonicity of Lagrangian submanifolds and they gave an elementary proof for non existence of type-I singularities and gave an extension result of Neves in his paper [35].

Kai Cieliebak and Edward Goldstein in [9] proved a simple relation between the mean curvature form, symplectic area, and the Maslov class of a Lagrangian immersion L in a Kähler-Einstein manifold (M, ω) whose Ricci curvature is a multiple of the metric by a real number λ. This relation is the following :

$$\lambda\omega(F) - \pi\mu(F) = H(\partial F) \tag{1.3}$$

where $F : \Sigma \to M$ is a smooth map from a compact oriented surface Σ to M whose (possibly empty) boundary $\partial F := F(\partial\Sigma)$ is contained in L, $\mu(F)$ is the Maslov class, $\omega(F) := \int_\Sigma F^*\omega$ and H is the mean curvature form of L. As an immediate consequence, minimal Lagrangian immersions are monotone in Kähler-Einstein manifolds with positive scalar curvature. The same relation (1.3) was given in [32] by Morvan for $\mathbb{C}^n$ and in [2] by Arsie for Calabi-Yau manifolds.

There are many other flows related directly or indirectly to mean curvature flow. In [7], Jingyi Chen and Jiayu Li consider compact symplectic surfaces moving by mean curvature flow in a Kähler-Einstein surface. They show that symplectic surfaces remain symplectic along the flow and that the flow does not develop any type-I singularities. J. Chen, J. Li and G. Tian in [8] showed that the flow has longtime existence in the graphical case and converges to a minimal surface. Yian Song and Ben Weinkove in their paper [48] prove some basic properties of Donaldson's flow of surfaces in a hyperKähler 4-manifold. They show that if the initial submanifold is symplectic with respect to one Kähler form and Lagrangian with respect to another, then certain kinds of singularities cannot form, and a convergence result follows under some condition.

In [52], Mu-Tao Wang considers the evolution of the graph of a smooth map $f : (\sum_1, g) \to (\sum_2, h)$ in $\sum_1 \times \sum_2$ by mean curvature flow where $(\sum_1, g)$ and $(\sum_2, h)$ are compact Riemannian manifolds of constant curvature k_1 and k_2 respectively. He showed that if $k_1 \geq k_2$ and $\det(g_{ij} + (f^*h)_{ij}) < 2$, then the mean curvature flow of the graph of f remains a graph and exists for all time. Moreover if in addition $k_1 + k_2 > 0$, the mean curvature flow of the graph of f converges to a graph of a constant map at infinity.

Bernstein's classical result says if the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is minimal, then f is linear. In their paper [54], Tsui, Mao-Pei and Wang Mu-Tao ask the following question : If Σ, the graph of $f : \mathbb{R}^n \to \mathbb{R}^m$, is minimal, what additional conditions on f force Σ to be planar? In codimension 1-case ($m = 1$), it is well known that Bernstein's result extends to $n \leq 7$ without additional hypothesis on f. Without restrictions on n, it also holds under certain growth conditions on f (e.g., $|df|$ bounded; see [12]). In case $n = m$, with the additional requirement that Σ be Lagrangian with respect to the standard symplectic form $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ on $\mathbb{R}^{2n}$. This condition implies that $f = \nabla F$ for some $F : \mathbb{R}^n \to \mathbb{R}$. They show that, if $\lambda_i \leq K$ and $\lambda_i \lambda_j \geq -1$, where λ_i are the eigenvalues of $D^2 F$, any special Lagrangian graph is planar. K. Smoczyk, G. Wang, and Y. L. Xin, also worked on Bernstein type theorems and they proved Bernstein type theorems for minimal n-submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle. Their results are natural generalizations of the corresponding results of Schoen-Simon-Yau and Ecker-Huisken for minimal hypersurfaces (see [12] and [37]).

There are many possible ways to get minimal submanifolds. We can get it by mean curvature flow, or Donaldson's flow [48], and other flows. In this thesis, we also try to get minimal Lagrangian submanifolds by deforming Lagrangian submanifolds by functions of their Lagrangian angle. There exist examples of similar situations in the literature. For example many people try to deform hypersurfaces by functions of their mean curvature. We can cite for example the work of Felix Schulze where he considers submanifolds deformed by a power of their mean curvature (cf. [38], [39] and references therein).

A special example is the inverse mean curvature flow which was used by Huisken and Ilmanen to settle the Riemannian Penrose conjecture and which has also been studied by many other authors([4], [15], [16], [18], [19], [22], [23], [24], [33], [34], [47], [49]).

In this thesis, we are interested in Lagrangian immersions L in Kähler-Einstein manifolds $(\bar{M}, \bar{g}, J)$ evolving by functions of their Lagrangian angle. More precisely, let

$$F_0 : L \to \bar{M}$$

be a Lagrangian immersion from a manifold L into $\bar{M}$. We consider a one-parameter family of immersions $F_t : L \to \bar{M}$ which evolves by :

$$\begin{aligned} \frac{dF}{dt} &= J\nabla f \\ F(\cdot, 0) &= F_0 \,, \end{aligned} \tag{1.4}$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a function (at least C^1) of the Lagrangian angle α, ∇f denotes the gradient of f considered as a function on L w.r.t. the Levi-Civita connection ∇ and J is the complex structure on $T\bar{M}$. In order to consider functions of the angle, we must assume that there exists a globally defined

Lagrangian angle. Usually there exists only a locally defined Lagrangian angle, i.e. a function α with $d\alpha = H$ giving the mean curvature 1-form $H = \bar{g}(J, \overrightarrow{H})$. α exists globally, iff H is exact which is equivalent to a trivial first Maslov class $m_1 = [H]/\pi$. In particular, this condition is satisfied for Lagrangian graphs over $\mathbb{R}^n$ in $\mathbb{C}^n$.

In case $f(\alpha) = \alpha$ we obtain the usual Lagrangian mean curvature flow. In this thesis we will consider a special class $\mathcal{F}_{a,b,\epsilon}$ of functions f for which we will prove short-time resp. long-time existence results and in the case of graphs in euclidean space we will be able to describe the limit behavior of $L_t := F(L, t)$ as t approaches the maximal time of existence T.

In general, for the mean curvature flow (1.1), one cannot expect longtime existence results without extra assumptions on the initial submanifold. In [42], Smoczyk considered Lagrangian submanifolds generated by symplectic maps and proved convergence to minimal Lagrangian maps under very natural and sharp conditions for the Lagrangian angle α. In our flow, we will prove that the flow will stay parabolic as long as the solution exists and we will give another sufficient condition for longtime existence that entirely differs from those conditions above but is similar to the conditions used in [43].

The organization of this thesis is as follows : In chapter 2, I summarize the most relevant facts and geometric equations for Lagrangian submanifolds in particular of Lagrangian submanifolds in Kähler-Einstein manifolds. I give some examples of Lagrangian submanifolds and reprove some well-known results for Lagrangian submanifolds like the full symmetry of the second fundamental form, the Gauss formula, Gauss equation, Codazzi equation, Ricci equation. I will also introduce and explain my notation in that chapter.

In chapter 3 I recall some basic methods from the theory of partial differential equations, like the maximum principle for tensors and for functions, etc. that are essential for the proofs of my theorems.

In chapter 4, I will discuss the main results of my thesis which are about the deformation of Lagrangian submanifolds in Kähler-Einstein manifolds by appropriate functions of their Lagrangian angle. Whereas some results are more general, most of my work focuses on the flow in $\mathbb{R}^{2n}$ or more generally in flat spaces. After stating my main theorem (Theorem 4.2) I will first derive the most relevant evolution equations for various geometric quantities, like the metric, volume form, second fundamental and mean curvature form and in particular the Lagrangian angle. After this I will show that the flow can be integrated to some underlying parabolic equation of Monge-Ampère type. Using the maximum principle for tensors I will derive the necessary a-priori estimates in C^2 in space and C^1 in time. Based on methods due to Krylov and Safonov I will then show a-priori estimates in space and time of the type $(C^{2,\alpha}, C^{1,\alpha})$. Finally, a Harnack inequality can be applied to a suitable function of the Lagrangian angle to obtain convergence of the solution to a flat graph. The thesis ends with some remarks concerning a monotonicity formula for this flow.

2

Lagrange submanifolds

In symplectic geometry there is a distinguished class of immersions, known as Lagrangian submanifolds. These are n-dimensional submanifolds L in $2n$-dimensional symplectic manifolds $(M, \overline{\omega})$ such that $\omega := \overline{\omega}_{TL} = 0$.
The most prominent examples of symplectic manifolds are Kähler manifolds $(M, J, \overline{g})$, where

$$\overline{\omega}(V, W) = \overline{g}(JV, W)$$

is the symplectic 2-form (Kähler form) induced by the Kähler metric $\overline{g}$ and the complex structure J.
Let L be a compact manifold and let $F_0 : L \longrightarrow M$ be a smooth Lagrangian immersion into a Kähler-Einstein manifold M.

The mean curvature form H of Lagrangian submanifolds in Kähler manifolds $(M, J, \overline{g})$ is related to $\overrightarrow{H}$ through

$$H(V) = \overline{\omega}(\overrightarrow{H}, V).$$

If $(M, J, \overline{g})$ is Kähler-Einstein, then H is closed and any locally defined function α with

$$d\alpha = H \tag{2.1}$$

is called a Lagrangian angle.

2.1 Examples of Lagrangian submanifolds

In this section we will give some standard examples of Lagrangian submanifolds and we will also recall some of their alternative descriptions.

1) **Curves in 2-dimensional manifolds.** This is the easiest examples of all. Any curve in 2-dimensional manifold is Lagrangian since a 2-form restricted to a 1-dimensional manifold must vanish.

2) **Graphs of symplectomorphisms.** Assume that $(M_1, \omega_1), (M_2, \omega_2)$ are two symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called symplectic if $f^*\omega_2 = \omega_1$. A symplectomorphism is a symplectic diffeomorphism. Now consider the symplectic manifold (M, ω), where $M = M_1 \times M_2$ and $\omega = \omega_1 \oplus -\omega_2$. A diffeomorphism $f : M_1 \to M_2$ generates a graph in M namely
$$\Gamma_f := \{(x, f(x)) : x \in M_1.\}$$
Then we can easily check that Γ_f is a Lagrangian immersion in (M, ω) if and only $f : (M_1, \omega_1) \to (M_2, \omega_2)$ is a symplectomorphism.

3) **Cotangent bundles.** Let N be a differentiable manifold. The cotangent bundle of N is given by
$$T^*N = \{\text{linear maps} f : T_qN \to \mathbb{R}, q \in N.\}$$
If $(q^1, \ldots, q^n)$ are local coordinates on $U \subset N$, then for a fixed $q \in U$ a 1-form $p_i dq^i$ on T_qN is defined by $(p_1, \ldots, p_n)$. This implies that local coordinates for an element $\bar{l} \in T^*N$ are given by $(q^1, \ldots, q^n, p_1, \ldots, p_n)$. $\bar{l} = pdq = p_i dq^i$ is called Liouville form. Now define a 1-form $\overline{\theta}$ on $M = T^*N$ by
$$\overline{\theta}(X) := \bar{l}(\pi_* X),$$
where $X \in T_{\bar{l}}(T^*N)$ and $\pi_* : TT^*N \to TN$ is the derivative of the natural projection. $\overline{\omega} := d\overline{\theta}$ (see [44]) is a well defined symplectic form on T^*N. Locally we have $\overline{\omega} = dp \wedge dq$. If $\eta \in \Omega^1(N)$ is a smooth 1-form on N, then η defines a graph in T^*N by sending $p \in N$ to $(p, \eta(p)) \in T^*N$. This graph is Lagrangian in $(T^*N, \overline{\omega})$ if and only if η is closed.

2.2 A local property of Lagrangian submanifolds

Given two symplectic manifolds (M_1, Ω_1) and (M_2, Ω_2), does there exist a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*\Omega_2 = \Omega_1$? Such a diffeomorphism is called a symplectomorphism (see above). The classification problem is still unsolved. However, a local classification is achieved by Darboux theorem which asserts that locally, all symplectic manifolds look alike. More precisely, we have the following

Theorem 2.1 (Darboux) *For any point x in the symplectic manifold (M, ω) there exists an open neighborhood U of x and a local chart $\varphi : U \to \mathbb{R}^{2n}$ with $\varphi(x) = 0$ and $\varphi^*\omega_s = \omega|_U$, where ω_s is the standard symplectic structure on $\mathbb{R}^{2n}$.*

Now consider $\mathbb{R}^n \subset \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ and assume that we are given n different height functions $u_1, \ldots, u_n$ on an open domain $\Omega \subset \mathbb{R}^n$, i.e.

$$u_i : \Omega \to \mathbb{R}$$

are smooth functions. The graph of $(u_1, \dots, u_n)$ is given by

$$\Gamma_u := \left\{(x^1, \dots, x^n, u_1, \dots, u_n) \subset \mathbb{C}^n, x \in \Omega\right\}.$$

It can be easily seen that Γ_u is Lagrangian in $\mathbb{C}^n$ if and only if the 1-form $u_i dx^i$ on Ω is closed. Thus if Ω is simply connected we can find a potential u such that $u_i = \frac{\partial u}{\partial x^i}$. Since every submanifold can locally be written as a graph, then by the theorem of Darboux 2.1, Lagrangian submanifolds can locally be written as gradient graphs over their tangent planes. Thus the Lagrangian condition can be understood as an integrability condition for the n height functions. This fact is very useful in the analysis of Lagrangian submanifolds, in particular we will exploit this in the analysis of the flows we consider in this thesis.

2.3 Notations

In this section we explain some of our notation. To begin, assume that $(\bar{M}, \bar{g}, J, \bar{\omega})$ is the Kähler manifold with compatible complex structure J and the Kähler form $\bar{\omega}$. Local coordinates on $\bar{M}$ will be denoted by $(y^A)_{A=1,\dots,2n}$ whereas local coordinates for a Lagrangian submanifold L will be denoted by $(x^i)_{i=1,\dots,n}$. Moreover, we use the Einstein convention to sum over repeated indices, the sum is taken from 1 to $2n$ for Latin capital indices and from 1 to n for Latin small letter indices and that an underlined small letter denotes the application of the complex structure J, e.g.

$$\overline{R}_{ABC\underline{D}} := \overline{R}\left(\frac{\partial}{\partial y^A}, \frac{\partial}{\partial y^B}, \frac{\partial}{\partial y^C}, J\frac{\partial}{\partial y^D}\right).$$

We will denote $\overline{\nabla}$ the Levi-Civita connection on $\bar{M}$, by $\overline{\Gamma}^A_{BC}$ his Christoffel symbols, $\overline{R}_{AB}$ the Ricci curvature, $\bar{g}_{AB}$ the metric, $\bar{\omega}_{AB}$ the Kähler form on $\bar{M}$, and we will set

$$\partial_A := \frac{\partial}{\partial y^A}$$

and

$$\overline{\Gamma}_{AB} := \overline{\Gamma}^C_{AB}\frac{\partial}{\partial y^C}.$$

Now, let

$$F : L \to \bar{M}$$

a Lagrangian immersion. We will set

$$\partial_i := \frac{\partial}{\partial x^i},$$

$$
\begin{aligned}
F_i &:= \frac{\partial F}{\partial x^i}, \\
\nu_i &:= JF_i, \\
F_{ij} &:= \frac{\partial^2 F}{\partial x^i \partial x^j}.
\end{aligned}
$$

By definition we have :

$$L_0 := F(L) \text{ is Lagrangian} :\Leftrightarrow \bar{\omega}^* := F^* \bar{\omega} = 0.$$

Then, this implies that ν_i is normal vector for each $i = 1, \cdots, n$. In the sequel we will often rise and lower indices w.r.t. the metric tensor g^{ij}, $g_{ij} = \bar{g}(F_i, F_j)$, for example $h^k{}_{ij} = g^{kl} h_{lij}$.

We will also denote ∇ the connection on the vector bundle $F^*(T\bar{M})$ over L.

The second fundamental tensor on L will be defined as :

$$h_{ijk} := \bar{g}(\nu_i, \nabla_{\partial_j} F_k)$$

and the mean curvature form is given by

$$H_i := g^{kl} h_{ikl}.$$

We will also introduce

$$
\begin{aligned}
A_{ijkl} &:= h_{ijn} h^n{}_{kl}, \\
a_{kl} &:= A_i{}^i{}_{kl} = H^n h_{nkl}, \\
b_{kl} &:= A_k{}^i{}_{li} = h_k{}^{ij} h_{ijl}.
\end{aligned}
$$

We have

$$\nabla_{\partial_i} F_j = (\nabla_{\partial_i} F_j)^\top + (\nabla_{\partial_i} F_j)^\perp$$

and we will set

$$\nabla^\top_{\partial_i} F_j := (\nabla_{\partial_i} F_j)^\top := \Gamma^k_{ij} F_k$$

and $\top$ means the tangential part of the submanifold and $\perp$ means the normal part of the submanifold.
$\nabla^\top$ is a connection on the tangent bundle of the submanifold, and setting

$$\nabla^\perp_{\partial_i} \nu_j := (\nabla_{\partial_i} \nu_j)^\perp$$

we define a connection $\nabla^\perp$ on the normal bundle NL of the submanifold.

The curvature on the vector bundle $F^*(T\bar{M})$ over L will be denoted R, the one of tangent bundle of the submanifold will be denoted $R^\top$, and the one of normal bundle of the submanifold will be denoted $R^\perp$.

We will also denote

$$R_{ijkl} := R^\top_{ijkl} = \bar{g}\left(R^\top(\partial_i, \partial_j) F_l, F_k\right)$$

and

$$R_{ij} = R^k{}_{ikj},$$

$$\overline{R}_{ijkl} := -\bar{g}\left(R(\partial_i, \partial_j)F_k, F_l\right) = -F_i^A F_j^B F_k^C F_l^D \overline{R}_{ABCD}.$$

We also write

$$\overline{\nabla}_i \overline{R}_{jklm} := F_i^A F_j^B F_k^C F_l^D F_m^E \overline{\nabla}_A \overline{R}_{BCDE},$$

$$\overline{R}_{ijk\underline{l}} := -\bar{g}\left(R(\partial_i, \partial_j)F_k, JF_l\right),$$

$$\overline{R}_{ij} := F_i^A F_j^B \overline{R}_{AB}.$$

2.4 Geometric identities for Lagrangian submanifolds

In this section we list some identities for Lagrangian submanifolds, in particular for Lagrangian submanifolds in Kähler-Einstein manifolds.

We have the well-known following equations:

Proposition 2.1.

Full symmetry of the second fundamental tensor :

$$h_{ijk} = h_{jik} = h_{jki} \ , \tag{2.2}$$

Gauss formula :

$$h_{jk}^n \nu_n = F_{kj} - \Gamma_{jk}^n F_n + F_j^A F_k^B \overline{\Gamma}_{AB} \ , \tag{2.3}$$

Gauss equation :

$$R_{ijkl}^{\top} = \overline{R}_{ijkl} + A_{ikjl} - A_{iljk} \ , \tag{2.4}$$

Codazzi equation :

$$\nabla_i h_{jkl} - \nabla_j h_{ikl} = \overline{R}_{ij\underline{k}l} \ , \tag{2.5}$$

Traced Codazzi equation :

$$\nabla_k H_l - \nabla_l H_k = \overline{R}_{k\underline{l}} \ , \tag{2.6}$$

Ricci equation :

$$\begin{aligned} R^{\perp}(\partial_i, \partial_j)\nu_k &= (R(\partial_i, \partial_j)\nu_k)^{\perp} - (A_{is} \wedge A_j^s)(\nu_k) \\ &= R_{ij}{}^s{}_k \nu_s. \end{aligned} \tag{2.7}$$

For the convenience of the reader we will include some proofs here.

Proof:

$$h_{ijk} := \bar{g}\left(\nu_i, \nabla_{\partial_j} F_k\right).$$

Now $\nabla_{\partial_j} F_k := F_{jk} + F_j^A F_k^B \overline{\Gamma}_{AB} = \nabla_{\partial_k} F_j$.

So,

$$h_{ijk} = h_{ikj}.$$

Also

$$h_{ijk} := \bar{g}(\nu_i, \nabla_{\partial_j} F_k) = -\bar{g}(\nabla_{\partial_j} \nu_i, F_k) = -\bar{g}\left(\nabla_{\partial_j} JF_i, F_k\right).$$

From $\overline{\nabla} J = 0$, we get

$$h_{ijk} = -\bar{g}\left(J\nabla_{\partial_j} F_i, F_k\right) = \bar{g}\left(\nabla_{\partial_j} F_i, JF_k\right) = \bar{g}\left(\nabla_{\partial_j} F_i, \nu_k\right) = h_{kji}.$$

We get then the full symmetry of the second fundamental form.

We have $\nabla_j F_k := \nabla_{\partial_j} F_k - \Gamma_{jk}^n F_n = (\nabla_{\partial_j} F_k)^{\perp} = h_{jk}^n \nu_n$.

Now, $\nabla_{\partial_j} F_k := F_{jk} + F_j^A F_k^B \overline{\Gamma}_{AB}$. We get then the Gauss formula.

To prove Gauss and Codazzi equations, we compute

$$\begin{aligned} R(\partial_i, \partial_j) F_k &:= \nabla_{\partial_i} \nabla_{\partial_j} F_k - \nabla_{\partial_j} \nabla_{\partial_i} F_k - \nabla_{[\partial_i, \partial_j]} F_k \\ &= \nabla_{\partial_i} \nabla_{\partial_j} F_k - \nabla_{\partial_j} \nabla_{\partial_i} F_k. \end{aligned}$$

But

$$\nabla_{\partial_j} F_k = (\nabla_{\partial_j} F_k)^{\top} + (\nabla_{\partial_j} F_k)^{\perp} = \nabla_{\partial_j}^{\top} F_k + h_{jk}^s \nu_s.$$

Then,

$$\begin{aligned} \nabla_{\partial_i} \nabla_{\partial_j} F_k &= \nabla_{\partial_i}(\nabla_{\partial_j}^{\top} F_k + h_{jk}^s \nu_s) \\ &= \nabla_{\partial_i}^{\top} \nabla_{\partial_j}^{\top} F_k + (\nabla_{\partial_i} \nabla_{\partial_j}^{\top} F_k)^{\perp} + (\nabla_{\partial_i}(h_{jk}^s \nu_s))^{\top} + \nabla_{\partial_i}^{\perp}(h_{jk}^s \nu_s) \\ &= \nabla_{\partial_i}^{\top} \nabla_{\partial_j}^{\top} F_k + (\nabla_{\partial_i}(\Gamma_{jk}^s F_s))^{\perp} + (\nabla_{\partial_i}(h_{jk}^s \nu_s))^{\top} + \nabla_{\partial_i}^{\perp}(h_{jk}^s \nu_s) \\ &= \nabla_{\partial_i}^{\top} \nabla_{\partial_j}^{\top} F_k + (\nabla_{\partial_i}(\Gamma_{jk}^s F_s))^{\perp} + (\nabla_{\partial_i}(h_{jk}^s \nu_s))^{\top} + \partial_i(h_{jk}^s)\nu_s \\ &\quad + h_{jk}^s \nabla_{\partial_i}^{\perp} \nu_s. \end{aligned}$$

Now,

$$\begin{aligned} \nabla_{\partial_i}^{\perp} \nu_s &= \nabla_{\partial_i}^{\perp} JF_s \\ &= (\nabla_{\partial_i} JF_s)^{\perp}. \end{aligned}$$

From $\overline{\nabla} J = 0$, we get

$$\begin{aligned} \nabla_{\partial_i}^{\perp} \nu_s &= (J\nabla_{\partial_i} F_s)^{\perp} \\ &= J\nabla_{\partial_i}^{\top} F_s \\ &= J(\Gamma_{is}^m F_m) \\ &= \Gamma_{is}^m \nu_m. \end{aligned} \tag{2.8}$$

So

$$
\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}F_k &= \nabla^\top_{\partial_i}\nabla^\top_{\partial_j}F_k + (\nabla_{\partial_i}(\Gamma^s_{jk}F_s))^\perp + (\nabla_{\partial_i}(h^s_{jk}\nu_s))^\top + \partial_i(h^s_{jk})\nu_s \\
&\quad + \Gamma^m_{is}h^s_{jk}\nu_m \\
&= \nabla^\top_{\partial_i}\nabla^\top_{\partial_j}F_k + g^{ml}\bar{g}\left(\nabla_{\partial_i}(\Gamma^s_{jk}F_s),\nu_m\right)\nu_l \\
&\quad + g^{ml}\bar{g}\left(\nabla_{\partial_i}(h^s_{jk}\nu_s),F_m\right)F_l + \nabla_i(h^s_{jk})\nu_s + \Gamma^m_{ij}h^s_{mk}\nu_s + \Gamma^m_{ik}h^s_{jm}\nu_s \\
&= \nabla^\top_{\partial_i}\nabla^\top_{\partial_j}F_k + g^{ml}\Gamma^s_{jk}h_{ism}\nu_l - g^{ml}h^s_{jk}h_{ism}F_l \\
&\quad + \nabla_i(h^s_{jk})\nu_s + \Gamma^m_{ij}h^s_{mk}\nu_s + \Gamma^m_{ik}h^s_{jm}\nu_s \\
&= \nabla^\top_{\partial_i}\nabla^\top_{\partial_j}F_k + \Gamma^m_{jk}h^s_{im}\nu_s - h^s_{jk}h^l_{is}F_l \\
&\quad + \nabla_i(h^s_{jk})\nu_s + \Gamma^m_{ij}h^s_{mk}\nu_s + \Gamma^m_{ik}h^s_{jm}\nu_s.
\end{aligned}
$$

So,

$$
\begin{aligned}
R(\partial_i,\partial_j)F_k &= R^\top(\partial_i,\partial_j)F_k \\
&\quad + (h^s_{ik}h^m_{js} - h^s_{jk}h^m_{is})F_m + \nabla_i(h^s_{jk})\nu_s - \nabla_j(h^s_{ik})\nu_s
\end{aligned}
$$

Taking the scalar product of the right and left hand of this latter equation with F_l, we get the Gauss equation. And making scalar product of the right and left hand of this latter equation with ν_l, we get Codazzi equation.
To prove the traced Codazzi equation, we use the Codazzi equation and we get :

$$g^{kl}(\nabla_i h_{jkl} - \nabla_j h_{ikl}) = g^{kl}\overline{R}_{ij\underline{k}l}\ .$$

That gives

$$\nabla_i H_j - \nabla_j H_i = g^{kl}\overline{R}_{ij\underline{k}l}.$$

Now by Kähler identity ($\overline{R}_{ABCD} = \overline{R}_{AB\underline{C}\underline{D}}$) we have $g^{kl}\overline{R}_{ij\underline{k}l} = \frac{1}{2}\overline{R}_{ij\underline{C}}{}^{C}$. We now use the well-known identity $\overline{R}_{AB\underline{C}}{}^{C} = 2\overline{R}_{A\underline{B}}$ and we get $g^{kl}\overline{R}_{ij\underline{k}l} = R_{i\underline{j}}$ and then we obtain the result.

To prove the Ricci equation, we compute

$$
\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}\nu_k &= \nabla_{\partial_i}\Big((\nabla_{\partial_j}\nu_k)^\top + \nabla^\perp_{\partial_j}\nu_k\Big) \\
&= \nabla_{\partial_i}(-h^s_{jk}F_s + \nabla^\perp_{\partial_j}\nu_k) \\
&= -\nabla^\top_{\partial_i}(h^s_{jk}F_s) - (\nabla_{\partial_i}(h^s_{jk}F_s))^\perp + (\nabla_{\partial_i}\nabla^\perp_{\partial_j}\nu_k)^\top + \nabla^\perp_{\partial_i}\nabla^\perp_{\partial_j}\nu_k \\
&= -\partial_i(h^s_{jk})F_s - \Gamma^m_{is}h^s_{jk}F_m - (\nabla_{\partial_i}(h^s_{jk}F_s))^\perp \\
&\quad + (\nabla_{\partial_i}\nabla^\perp_{\partial_j}\nu_k)^\top + \nabla^\perp_{\partial_i}\nabla^\perp_{\partial_j}\nu_k \\
&\overset{(2.8)}{=} -\nabla_i h^s_{jk}F_s - \Gamma^m_{ij}h^s_{mk}F_s - \Gamma^m_{ik}h^s_{jm}F_s \\
&\quad - g^{ml}\bar{g}\left(\nabla_{\partial_i}(h^s_{jk}F_s),\nu_m\right)\nu_l \\
&\quad + g^{ml}\bar{g}\left(\nabla_{\partial_i}(\Gamma^s_{jk}\nu_s),F_m\right)F_l + \nabla^\perp_{\partial_i}\nabla^\perp_{\partial_j}\nu_k \\
&= -\nabla_i h^s_{jk}F_s - \Gamma^m_{ij}h^s_{mk}F_s - \Gamma^m_{ik}h^s_{jm}F_s \\
&\quad - h^s_{jk}h^l_{is}\nu_l - \Gamma^s_{jk}h^l_{is}F_l + \nabla^\perp_{\partial_i}\nabla^\perp_{\partial_j}\nu_k.
\end{aligned}
$$

So,

$$\begin{aligned} R(\partial_i,\partial_j)\nu_k &:= \nabla_{\partial_i}\nabla_{\partial_j}\nu_k - \nabla_{\partial_j}\nabla_{\partial_i}\nu_k - \nabla_{[\partial_i,\partial_j]}\nu_k \\ &= \nabla_{\partial_i}\nabla_{\partial_j}\nu_k - \nabla_{\partial_j}\nabla_{\partial_i}\nu_k \\ &= R^{\perp}(\partial_i,\partial_j)\nu_k - (\nabla_i h^s_{jk} - \nabla_j h^s_{ik})F_s \\ &\quad -(h^s_{jk}h^l_{is} - h^s_{ik}h^l_{js})\nu_l. \end{aligned}$$

Then,

$$R^{\perp}(\partial_i,\partial_j)\nu_k = (R(\partial_i,\partial_j)\nu_k)^{\perp} + (h^s_{jk}h^l_{is} - h^s_{ik}h^l_{js})\nu_l.$$

q.e.d.

3

Methods and auxiliary material

3.1 Maximum principle

The maximum principle has proved to be a powerful tool in partial differential equations. In particular, the maximum principle of parabolic systems for tensors developed by R. Hamilton ([21]) plays an important role in the study of geometric evolution equations. The theorem is the following :
Let u^k be a vector field and let g_{ij}, M_{ij} and N_{ij} be symmetric tensors on a compact manifold M which may depend on time t and g_{ij} is the metric tensor on M. Assume that $N_{ij} = p(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using the metric. Furthermore, let this polynomial satisfy a null-eigenvector condition, i.e. for any null-eigenvector X of M_{ij} we have $N_{ij}X^iX^j \geq 0$. Then we have

Theorem 3.1 (Hamilton) *Suppose that on* $0 \leq t < T$ *the evolution equation*

$$\frac{\partial}{\partial t} M_{ij} = \triangle M_{ij} + u^k \nabla_k M_{ij} + N_{ij}$$

holds, where $N_{ij} = p(M_{ij}, g_{ij})$ *satisfies the null-eigenvector condition above. If* $M_{ij} \geq 0$ *at* $t = 0$*, then it remains so on* $0 \leq t < T$*.*

We will also repeatedly use the strong and weak parabolic and elliptic maximum principle for functions on compact manifolds. In the non-compact case the following maximum principle for the mean curvature flow has been proved by Ecker and Huisken

Theorem 3.2 (Ecker-Huisken) *Suppose the function* $f : M \times [0, T) \to \mathbb{R}$ *satisfies the inequality*

$$\left(\frac{d}{dt} - \triangle \right) f \leq g(a, \nabla f)$$

for some a which is uniformly bounded on $M \times [0, t_1]$ *for some* $t_1 > 0$, *then*

$$\sup_{M_t} f \leq \sup_{M_0} f$$

for all $t \in [0, t_1]$.

The proof of this theorem is based on the monotonicity formula for mean curvature flow [11].

3.2 Miscellaneous results

The following theorem is well-known:

Theorem 3.3 *There are no minimal closed submanifolds in* $\mathbb{R}^m$.

There is a very simple proof of this, namely

Proof: Let $F : L^n \to \mathbb{R}^m$ be an immersion and L closed. We have $\Delta|F|^2 = \nabla^i\nabla_i|F^2| = \nabla^i(2\langle F_i, F\rangle) = 2\langle H, F\rangle + 2n$, where H is the mean curvature vector and $\langle\cdot,\cdot\rangle$ is the euclidean metric. So if L is minimal ($H = 0$), we get $\Delta|F|^2 = 2n$ and we obtain a contradiction by the strong elliptic maximum principle. *q.e.d.*

In [42], Smoczyk proved the following longtime existence and convergence result for the Lagrangian mean curvature flow in Kähler-Einstein manifold of nonpositive scalar curvature. We will later use the same technique in the proof of this theorem to prove a similar result for our flow which will be very useful to prove our main theorem 4.2.

Proposition 3.1. *Let* L *be a compact manifold and* $F_0 : L \longrightarrow M$ *be a smooth Lagrangian immersion into a Kähler-Einstein manifold* $(M, J, \overline{g})$ *that is either compact or complete with bounded curvature quantities. Further assume that* $[0, T)$, $0 < T \leq \infty$ *is the maximal time interval on which the Lagrangian mean curvature flow admits a smooth solution. Then the following is true:*
(a) Assume there exists a constant $C_0 < \infty$ *such that*

$$\max_{L_t} |A|^2 \leq C_0, \forall t \in [0, T),$$

where $|A|^2$ *is the squared norm of the second fundamental tensor* A. *Then for any* $m \geq 0$ *there exists a constant* $C_m < \infty$ *depending on* m, L_0, M *such that*

$$\max_{L_t} |\nabla^m A|^2 \leq C_m, \forall t \in [0, T).$$

(b) If $T < \infty$, *then*

$$\limsup_{t\to T}\max_{L_t}|A|^2=\infty.$$

(c) If in addition to (a) the initial mean curvature form of L_0 is exact, the ambient Kähler-Einstein manifold has non-positive Ricci curvature and the induced Riemannian metrics $F_t^*\overline{g}$ on L are all uniformly equivalent, then $T=\infty$ and the Lagrangian submanifolds L_t converge smoothly and exponentially to a smooth compact minimal Lagrangian immersion $L_\infty\subset M$.

4

Deforming Lagrangian submanifolds by functions of their angle

Let

$$F_0 : L \mapsto \bar{M}$$

be a Lagrangian immersion from a n-dimensional manifold L into a Kähler-Einstein manifold $(\bar{M}, \bar{g}, J)$ with metric $\bar{g}$ and complex structure J and suppose that the first Maslov class $m_1 = [H]/\pi = 0$, where $H = \bar{g}(J\cdot, \overrightarrow{H})$ denotes the mean curvature 1-form on L_0 and $\overrightarrow{H}$ is the mean curvature vector field along L_0. From the Codazzi equation we obtain that the mean curvature form is closed if L_0 is a Lagrangian submanifold in a Kähler-Einstein manifold and if $m_1 = 0$, we get a globally defined function α (unique up to adding a constant) with $d\alpha = H$. α is called the Lagrangian angle of L_0. We are looking for a one-parameter family of immersions $F_t : L \mapsto \bar{M}$ which evolves by :

$$\begin{aligned} \frac{dF}{dt} &= J\nabla f \\ F(\cdot, 0) &= F_0\,, \end{aligned} \tag{4.1}$$

where we assume that $m_1(t) = \frac{1}{\pi}[H(\cdot, t)] = 0$ which means that $H(\cdot, t) = d\alpha(\cdot, t)$ for every t where the solution of (4.1) exists. The normalization of $\alpha_t := \alpha(\cdot, t)$ is $\alpha_0 = \alpha$ and $f : \mathbb{R} \mapsto \mathbb{R}$ is a function (at least C^1) of the Lagrangian angle α_t and ∇f denotes the gradient of f considered as a function on L w.r.t. the Levi-Civita connection ∇.

(4.1) is equivalent to

$$\begin{aligned} \frac{dF}{dt} &= f'\overrightarrow{H} = f' \operatorname{tr}(\nabla dF) \\ F(\cdot, 0) &= F_0\,, \end{aligned} \tag{4.2}$$

where the last term is the trace of the second fundamental form $A = \nabla dF$.

Before we will be able to formulate our main theorems, we need to discuss a special class of endomorphisms on $\mathbb{R}^{2n}$ that will be essential in the analysis of the flow in the case of graphs.

4.1 Semilinear involutions

In this section we will (like in [43]) consider a class $\mathscr{C}$ of endomorphisms on $\mathbb{R}^{2n}$ that satisfy certain conditions. We will say $\overline{S} \in \mathrm{End}(\mathbb{R}^{2n})$ belongs to $\mathscr{C}$, if the following is true:

$$\overline{S}J = -J\overline{S}\,, \tag{4.3}$$

$$\overline{S}^t = \overline{S}\,, \tag{4.4}$$

$$\overline{S}^2 = \mathrm{Id}\,. \tag{4.5}$$

Here $J \in \mathrm{End}(\mathbb{R}^{2n})$ denotes the standard complex structure on $\mathbb{R}^{2n}$. Hence $\overline{S} \in \mathscr{C}$ are the self-adjoint, semi-linear involutions of $\mathbb{R}^{2n} = \mathbb{C}^n$.

Example 4.1. The standard example for a tensor $\overline{S} \in \mathscr{C}$ is complex conjugation

$$\overline{S}V := \overline{V}\,, \quad V \in \mathbb{R}^{2n} = \mathbb{C}^n\,.$$

Lemma 4.2. *The complex structure J operates on $\mathscr{C}$ by*

$$\overline{S} \mapsto \overline{S}^* := \overline{S}J\,.$$

Moreover

$$\overline{S}\,\overline{S}^* = J = -\overline{S}^*\,\overline{S}\,.$$

Proof:

$$(\overline{S}^*)^2 = \overline{S}J\overline{S}J = -\overline{S}^2J^2 = -J^2 = \mathrm{Id}\,,$$

$$\overline{S}^*J = \overline{S}JJ = -J\overline{S}J = -J\overline{S}^*$$

and

$$(\overline{S}^*)^t = (\overline{S}J)^t = J^t\overline{S}^t = -J\overline{S} = \overline{S}J = \overline{S}^*\,.$$

Moreover,

$$\overline{S}\,\overline{S}^* = \overline{S}^2J = J$$

proves the last statement. *q.e.d.*

Lemma 4.3. *Let $\overline{S} \in \mathscr{C}$ be as above. Suppose*

$$F : L \to \mathbb{R}^{2n}$$

is a Lagrangian immersion and define

$$S \in \operatorname{End}(TL)\,, \quad SV := \left(\overline{S}V\right)^{\top}\,, \quad \forall\, V \in TL\,,$$
$$S^* \in \operatorname{End}(TL)\,, \quad S^*V := \left(\overline{S}^*V\right)^{\top}\,, \quad \forall\, V \in TL\,,$$

where $W^{\top}$ denotes the orthogonal projection of $W \in \mathbb{R}^{2n}$ onto the tangent space of L. Then

$$[S, S^*] = 0\,, \tag{4.6}$$
$$S^2 + (S^*)^2 = \mathrm{Id}\,. \tag{4.7}$$

Proof: Suppose $p \in L$ is a fixed point on a Lagrangian immersion in $\mathbb{R}^{2n}$. We choose an orthonormal basis $\{e_1, \ldots, e_n\} \in T_pL$. We set $\nu_k := Je_k \in T_p^{\perp}L$ (by the Lagrangian condition). Then for any vector $V \in T_pL$ we have

$$\begin{aligned} SV &= \overline{S}V - (\overline{S}V)^{\perp} \\ &= \overline{S}V - \sum_{k=1}^{n}\langle \overline{S}V, \nu_k\rangle \nu_k \\ &= \overline{S}V + \sum_{k=1}^{n}\langle J\overline{S}V, e_k\rangle \nu_k \\ &\overset{(4.3)}{=} \overline{S}V - \sum_{k=1}^{n}\langle \overline{S}^*V, e_k\rangle \nu_k \\ &= \overline{S}V - J\left(\sum_{k=1}^{n}\langle \overline{S}^*V, e_k\rangle e_k\right) \\ &= \overline{S}V - J\left(\sum_{k=1}^{n}\langle S^*V, e_k\rangle e_k\right) \\ &= \overline{S}V - J\,(S^*V)\,. \end{aligned} \tag{4.8}$$

From (4.8) we obtain

$$\begin{aligned} S^*SV &= (\overline{S}^*\, SV)^{\top} \\ &= (\overline{S}^*\,\overline{S}V - \overline{S}^*JS^*V)^{\top} \\ &= (-JV - \overline{S}J^2S^*V)^{\top} \\ &= (\overline{S}S^*V)^{\top} \\ &= SS^*V \end{aligned}$$

and

$$\begin{aligned} S^2V &= (\overline{S}SV)^\top \\ &= (\overline{S}^2V - \overline{S}JS^*V)^\top \\ &= (V - \overline{S}^*S^*V)^\top \\ &= V - (S^*)^2V\,. \end{aligned}$$

Hence

$$[S, S^*] = 0 \quad \text{and} \quad S^2 + (S^*)^2 = \mathrm{Id}\ .$$

q.e.d.

4.2 The flow equation

In the sequel we will consider a special class $\tilde{\mathcal{F}}_{a,b,\epsilon}$ of functions $\tilde{f}$.

Definition 4.4. *Suppose $\tilde{f} : (a,b) \to \mathbb{R}$ is a smooth function on some open interval $(a,b) \subset \mathbb{R}$ and let $1 > \epsilon > \frac{n}{\sqrt{n^2+4}}$ be fixed where n is the dimension of L. We will say $\tilde{f}$ belongs to $\tilde{\mathcal{F}}_{a,b,\epsilon}$, if the following holds:*

$$\tilde{f}'(\alpha) > 0\,, \quad \forall\, \alpha \in (a,b), \tag{4.9}$$

$$\tilde{f}''(\alpha) \leq -f'(\alpha)\,, \quad \forall\, \alpha \in (a,b), \tag{4.10}$$

$$\frac{2\epsilon}{n}\tilde{f}'(\alpha) + \tilde{f}''(\alpha)\sqrt{1-\epsilon^2} \geq 0\,, \quad \forall\, \alpha \in (a,b), \tag{4.11}$$

Example 4.5. As example, we can give

1. $$\tilde{f}(\alpha) := \ln(\alpha) \quad \text{with} \quad \alpha \in (a,b) := \left(\frac{n}{2\epsilon}\sqrt{1-\epsilon^2}, 1\right)\ .$$

2. $$\tilde{f}(\alpha) := -Ae^{-\alpha} + B, \quad \text{where} \quad A, B \quad \text{are constants} \quad \text{and} \quad A > 0.$$

The next theorem states that there is short-time existence of smooth solutions of $\frac{d}{dt}F = f'\overrightarrow{H}$ with $f = \tilde{f} \circ \alpha$ with $\tilde{f}$ satisfying (4.9), if the initial Maslov class is trivial and the angle α lies in the interval (a,b).

Theorem 4.1 *Let $L_0 \subset \bar{M}$ be a compact Lagrangian immersion and suppose L_0 has trivial Maslov class so that there is a globally defined Lagrangian angle α_0 on L_0. Let $\tilde{f}$ be a function that satisfies (4.9) on (a,b) and suppose $\alpha_0(p) \in (a,b)$ for all $p \in L_0$ Then the evolution equation (4.1) with $f = \tilde{f} \circ \alpha$ has a unique smooth solution L_t for a short time $(0,T)$, $T > 0$ and L_t is Lagrangian.*

Proof: We observe that as long as $f' := \tilde{f}' \circ \alpha$ is positive, (4.1) is a quasi-linear parabolic system. Then the statement follows from the standard theory of parabolic equations. Since L_0 is Lagrangian and $J\nabla f$ is the Hamiltonian vector field of the function f, the flow is indeed a Hamiltonian deformation of the initial Lagrangian immersion and it is well known that this preserves the Lagrangian condition. *q.e.d.*

Later we will see that (4.1) actually becomes uniformly parabolic in euclidean space, if $\tilde{f}$ satisfies (4.9) , $\alpha \in (a, b)$ and L is compact. More precisely, from the evolution equation for the Lagrangian angle (4.21) in Lemma 4.12 and from the maximum principle we can see that, if

$$\alpha(0, x) \in [\tilde{a}, \tilde{b}] \subset (a, b) ,$$

then for every $t > 0$, we have also $\alpha(t, x) \in [\tilde{a}, \tilde{b}]$ and thus $f' = \tilde{f} \circ \alpha$ stays uniformly bounded from above and below by some positive constants.

In general we do not expect longtime existence and in fact we may easily construct counterexamples but under some additional assumptions we will later derive longtime existence and convergence results for these flows, i.e. we will prove the following main theorem:

Theorem 4.2 *Let $F_0 : L \to \mathbb{R}^{2n}$ be a compact Lagrangian submanifold in $\mathbb{R}^{2n}$ such that its universal cover $\widetilde{L}$ is $\mathbb{R}^n$ immersed as a Lagrangian graph into $\mathbb{R}^{2n}$ and such that the Lagrangian angle α lies in the interval (a, b). Suppose there exists an $\epsilon \in (\frac{n}{\sqrt{n^2+4}}, 1)$ and a tensor $\overline{S} \in \mathscr{C}$ as in Section 4.1 with*

$$S(V, V) := F_0^* \overline{S}(V, V) > \epsilon |V|^2 \qquad \forall V \in TL \subset \mathbb{R}^{2n}, V \neq 0.$$

Then for any $\tilde{f} \in \mathcal{F}_{a,b,\epsilon}$ the flow $\frac{d}{dt}F = f'\overrightarrow{H}$ with $f' := \tilde{f}' \circ \alpha$ has the following properties:

a) All induced metrics $g_t := F_t^*(<, >)$ are uniformly equivalent to the initial metric g_0.

b) The flow has a longtime existence ($T = \infty$) and the Lagrangian submanifolds converge smoothly to a flat Lagrangian submanifold.

4.3 Evolution equations

To start a more detailed analysis of (4.1) we will now derive several evolution equations for geometrically reasonable quantities, like the induced metric, volume form and second fundamental form.

Lemma 4.6. *The induced metrics g_{ij} satisfies the evolution equations*

$$\frac{\partial}{\partial t} g_{ij} = -2f' a_{ij}. \tag{4.12}$$

Proof: We use normal coordinate system at a fixed point $F(p) \in \bar{M}$ to compute

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left(\bar{g}(F_i, F_j)\right)\\ &= \bar{g}\left(\frac{\partial}{\partial t} F_i, F_j\right) + \bar{g}\left(F_i, \frac{\partial}{\partial t} F_j\right).\end{aligned}$$

Now let us compute $\frac{\partial}{\partial t} F_i$. We have :

$$\begin{aligned}\frac{\partial}{\partial t} F_i &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial t} F\\ &= \frac{\partial}{\partial x^i}(f' H^l \nu_l)\\ &= \nabla_{\partial_i}(f' H^l \nu_l) - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &= \nabla^{\perp}_{\partial_i}(f' H^l \nu_l) + (\nabla_{\partial_i}(f' H^l \nu_l))^{\top} - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &= \nabla_i (f' H^l)\nu_l + g^{sm}\left\langle \nabla_{\partial_i}(f' H^l \nu_l), F_s\right\rangle F_m - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &= f'\nabla_i H^l \nu_l + f'' H_i H^l \nu_l - g^{sm} f' H^l \left\langle \nu_l, \nabla_{\partial_i} F_s\right\rangle F_m - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &= f'\nabla_i H^l \nu_l + f'' H_i H^l \nu_l - f' H^l h_{il}{}^m F_m - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &= f'\nabla_i H^l \nu_l + f'' H_i H^l \nu_l - f' a_i{}^s F_s - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}. \qquad (4.13)\end{aligned}$$

Then in a double normal coordinate system at a fixed point $(p, F(p)) \in L \times \overline{M}$, we have

$$\frac{\partial}{\partial t} F_i = f'\nabla_i H^l \nu_l + f'' H_i H^l \nu_l - f' a_i{}^s F_s.$$

So we get

$$\frac{\partial}{\partial t} g_{ij} = -2f' a_{ij}.$$

q.e.d.

Lemma 4.7. *The second fundamental form satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t} h_{ijk} = &-f' a_i{}^s h_{jks} + f'' H_k \nabla_j H_i + f' \nabla_k \nabla_j H_i\\ &+ f^{(3)} H_k H_j H_i + f'' \nabla_k H_j H_i\\ &+ f'' \nabla_k H_i H_j - f' a_j^s h_{ksi} - f' H^s \overline{R}_{\underline{i}jk\underline{s}}.\end{aligned}$$

Proof: We use normal coordinate system at a fixed point $F(p) \in \bar{M}$ to compute

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} &= \frac{\partial}{\partial t}\bar{g}(\nu_i, \nabla_{\frac{\partial}{\partial x^j}} F_k)\\ &= \bar{g}(\frac{\partial}{\partial t}\nu_i, \nabla_{\frac{\partial}{\partial x^j}} F_k) + \bar{g}(\nu_i, \frac{\partial}{\partial t}\nabla_{\frac{\partial}{\partial x^j}} F_k).\end{aligned}$$

Then, in a double normal coordinate system at a fixed point $(p, F(p)) \in L\times\bar{M}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_{\frac{\partial}{\partial x^j}} F_k &= \frac{\partial}{\partial t}(F_{jk} + F_j^A F_k^B \overline{\Gamma}_{AB})\\ &= \frac{\partial}{\partial t}F_{jk} + f' F_j^A F_k^B H^s \nu_s^C \partial_C \overline{\Gamma}_{AB}.\end{aligned}$$

From (4.13), we compute

$$\begin{aligned}\frac{\partial}{\partial t}F_{ij} &= \frac{\partial}{\partial x^j}\frac{\partial}{\partial t}F_i\\ &= \frac{\partial}{\partial x^j}(f'\nabla_i H^l \nu_l + f'' H_i H^l \nu_l - f' {a_i}^s F_s - f' F_i^A H^l \nu_l^B \overline{\Gamma}_{AB})\\ &= f'' H_j \nabla_i H^l \nu_l + f'(\nabla_j \nabla_i H^l - \Gamma_{jk}^l \nabla_i H^k + \Gamma_{ij}^k \nabla_k H^l)\nu_l\\ &\quad + f'(-{h_{jl}}^s \nabla_i H^l F_s - \nabla_i H^l F_j^A \nu_l^C \overline{\Gamma}_{AC} + \Gamma_{jl}^s \nabla_i H^l \nu_s)\\ &\quad + f^{(3)} H_j H_i H^l \nu_l + f'' \nabla_j H_i H^l \nu_l + f'' \Gamma_{ij}^s H_s H^l \nu_l\\ &\quad + f'' \nabla_j H^l H_i \nu_l - f'' \Gamma_{js}^l H_i H^s \nu_l - f'' {h_{jl}}^s H_i H^l F_s\\ &\quad - f'' F_j^A \nu_l^B H_i H^l \overline{\Gamma}_{AB} + f'' \Gamma_{jl}^s H_i H^l \nu_s - f'' H_j {a_i}^s F_s\\ &\quad - f'(\nabla_j a_i^s + \Gamma_{ij}^m a_m^s - \Gamma_{jm}^s a_i^m) F_s - f' a_i^s F_{js} - f'' H_j F_i^A H^l \nu_l^B \overline{\Gamma}_{AB}\\ &\quad - f'(F_{ij}^A H^l \nu_l^B + F_i^A \nu_l^B \nabla_j H^l\\ &\quad - F_i^A \nu_l^B H^s \Gamma_{js}^l - a_j^s F_i^A F_s^B - F_i^A F_j^C \nu_l^D H^l \overline{\Gamma}_{CD}^B + H^l F_i^A \nu_s^B \Gamma_{jl}^s)\overline{\Gamma}_{AB}\\ &\quad - f' F_i^A F_j^C H^l \nu_l^B \partial_C \overline{\Gamma}_{AB}.\end{aligned}$$

So,

$$
\begin{aligned}
\frac{\partial}{\partial t}F_{ij} = & (f''H_j\nabla_iH^l + f'\nabla_j\nabla_iH^l \\
& + f'\Gamma_{ij}^k\nabla_kH^l + f^{(3)}H_jH_iH^l \\
& + f''\nabla_jH_iH^l + f''\Gamma_{ij}^sH_sH^l + f''\nabla_jH^lH_i)\nu_l \\
& - (f'{h_{jl}}^s\nabla_iH^l + f''{h_{jl}}^sH_iH^l + f''H_j{a_i}^s \\
& + f'(\nabla_ja_i^s + \Gamma_{ij}^m a_m^s - \Gamma_{jm}^s a_i^m))F_s \\
& - f'a_i^sF_{js} - f''H_iF_j^AH^l\nu_l^B\overline{\Gamma}_{AB} - f''H_jF_i^AH^l\nu_l^B\overline{\Gamma}_{AB} \\
& - f'\nabla_iH^lF_j^A\nu_l^B\overline{\Gamma}_{AB} \\
& - f'(F_{ij}^AH^l\nu_l^B + F_i^A\nu_l^B\nabla_jH^l \\
& - F_i^A\nu_l^BH^s\Gamma_{js}^l - a_j^sF_i^AF_s^B - F_i^AF_j^C\nu_l^DH^l\overline{\Gamma}_{CD}^B + H^lF_i^A\nu_s^B\Gamma_{jl}^s)\overline{\Gamma}_{AB} \\
& - f'F_i^AF_j^CH^l\nu_l^B\partial_C\overline{\Gamma}_{AB}.
\end{aligned}
$$

And then in double normal coordinate system at a fixed point $(p, F(p)) \in L \times \bar{M}$

$$
\begin{aligned}
\frac{\partial}{\partial t}F_{ij} = & (f''H_j\nabla_iH^l + f'\nabla_j\nabla_iH^l + f^{(3)}H_jH_iH^l \\
& + f''\nabla_jH_iH^l + f''\nabla_jH^lH_i - f'a_i^s{h_{js}}^l)\nu_l \\
& - (f'{h_{jl}}^s\nabla_iH^l + f''{h_{jl}}^sH_iH^l \\
& + f''H_j{a_i}^s + f'\nabla_ja_i^s)F_s \\
& - f'F_i^AF_j^CH^l\nu_l^B\partial_C\overline{\Gamma}_{AB}.
\end{aligned}
$$

From (4.13) we derive:

$$
\begin{aligned}
\frac{\partial}{\partial t}\nu_i & = \frac{\partial}{\partial t}JF_i \\
& = \frac{\partial}{\partial t}(J_B^AF_i^B\partial_A) \\
& = \frac{\partial}{\partial t}(J_B^AF_i^B)\partial_A \\
& = \partial_CJ_B^A\frac{\partial}{\partial t}F^CF_i^B\partial_A + J_B^A(f'\nabla_iH^l\nu_l^B + f''H_iH^l\nu_l^B \\
& \quad - f'{a_i}^sF_s^B - f'F_i^DH^l\nu_l^C\overline{\Gamma}_{DC}^B)\partial_A \\
& = f'(\overline{\nabla}_CJ_B^A + J_D^A\overline{\Gamma}_{BC}^D - J_B^E\overline{\Gamma}_{CE}^A)H^l\nu_l^CF_i^B\partial_A - f'\nabla_iH^lF_l \\
& \quad - f''H_iH^lF_l - f'{a_i}^s\nu_s - f'J_B^AF_i^DH^l\nu_l^C\Gamma_{DC}^B\partial_A \\
& = -f'H^l\nu_l^CF_i^BJ_B^E\overline{\Gamma}_{CE} - f'\nabla_iH^lF_l \\
& \quad - f''H_iH^lF_l - f'{a_i}^s\nu_s.
\end{aligned}
$$

Then in a double normal coordinate system at a fixed point $(p, F(p)) \in L \times \bar{M}$, we have

$$\frac{\partial}{\partial t}\nu_i = -f'\nabla_i H^l F_l - f'' H_i H^l F_l - f' a_i{}^s \nu_s$$

Then :

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} = &-f' a_i{}^s h_{jks} + f'' H_k \nabla_j H_i + f' \nabla_k \nabla_j H_i \\ &+ f^{(3)} H_k H_j H_i + f'' \nabla_k H_j H_i \\ &+ f'' \nabla_k H_i H_j - f' a_j^s h_{ksi} - f' F_j^A F_k^B H^s \nu_s^C \bar{g}(\nu_i, \partial_B \overline{\Gamma}_{AC} - \partial_C \overline{\Gamma}_{BA}).\end{aligned}$$

Now in normal coordinates, we have

$$\begin{aligned}\partial_B \overline{\Gamma}_{AC} - \partial_C \overline{\Gamma}_{BA} &= \partial_B \overline{\nabla}_{\partial_A} \partial_C - \partial_C \overline{\nabla}_{\partial_B} \partial_A \\ &= \overline{\nabla}_{\partial_B} \overline{\nabla}_{\partial_A} \partial_C - \overline{\nabla}_{\partial_C} \overline{\nabla}_{\partial_B} \partial_A \\ &= \overline{\nabla}_{\partial_B} \overline{\nabla}_{\partial_C} \partial_A - \overline{\nabla}_{\partial_C} \overline{\nabla}_{\partial_B} \partial_A \\ &= \overline{R}(\partial_B, \partial_C) \partial_A \\ &= \overline{R}^D{}_{ABC} \partial_D.\end{aligned}$$

q.e.d.

Lemma 4.8. *The mean curvature form satisfies the evolution equation*

$$\frac{\partial}{\partial t} H_j = \nabla_j (f' d^\dagger H) + f' \frac{\overline{R}}{2n} H_j + \nabla_j (f'' |H|^2)$$

where $\overline{R}$ is the scalar curvature of $\bar{M}$ and $d^\dagger H := \nabla^i H_i$.

Proof: From lemma 4.7 we obtain :

$$
\begin{aligned}
\frac{\partial}{\partial t} H_j &= \frac{\partial}{\partial t}(g^{ik} h_{ijk}) \\
&= -g^{is} g^{mk} \frac{\partial}{\partial t} g_{sm} h_{ijk} + g^{ik}(-f' a_i{}^s h_{jks} + f'' H_k \nabla_j H_i \\
&\quad + f' \nabla_k \nabla_j H_i + f^{(3)} H_k H_j H_i + f'' \nabla_k H_j H_i \\
&\quad + f'' \nabla_k H_i H_j - f' a_j^s h_{ksi} - f' H^s \overline{R}_{\underline{i}\underline{j}k\underline{s}}) \\
&= f' a^{ik} h_{ijk} - f' a_j{}^s H_s + f'' H^i \nabla_j H_i \\
&\quad + f' \nabla_i \nabla_j H^i + f^{(3)} H_j |H|^2 \\
&\quad + f'' H^i \nabla_i H_j + f'' H_j d^\dagger H - f' H^s \overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{s} \\
&\overset{(4.16)}{=} f' a^{ik} h_{ijk} - f' a_j{}^s H_s + f'' H^i \nabla_j H_i \\
&\quad + f' \nabla_j \nabla_i H^i + f' H^s R^i{}_{sij} + f^{(3)} H_j |H|^2 \\
&\quad + f'' H^i \nabla_i H_j + f'' H_j d^\dagger H - f' H^s \overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{s} \\
&= f' a^{ik} h_{ijk} - f' a_j{}^s H_s + f'' H^i \nabla_j H_i \\
&\quad + f' \nabla_j d^\dagger H + f' H^s R_{sj} + f^{(3)} H_j |H|^2 \\
&\quad + f'' H^i \nabla_i H_j + f'' H_j d^\dagger H - f' H^s \overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{s}.
\end{aligned}
$$

But, by the first Bianchi identity, we obtain

$$-\overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{\underline{s}} = \overline{R}_{\underline{i}}{}^{i}{}_{\underline{s}j} + \overline{R}_{\underline{i}\underline{s}\underline{j}}{}^{i}$$

so

$$-\overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{\underline{s}} = \overline{R}_{\underline{s}\underline{j}\underline{i}}{}^{i} + \overline{R}_{\underline{i}\underline{s}\underline{j}}{}^{i}.$$

Since $\bar{g}(F_i, F_j) = \bar{g}(\nu_i, \nu_j)$ and the curvature operator is of type-$(1,1)$ then

$$2\overline{R}_{\underline{s}j\underline{i}}{}^{i} = \overline{R}_{\underline{s}j\underline{i}}{}^{i} + \overline{R}_{\underline{s}j\underline{i}}{}^{i} = \overline{R}_{\underline{s}j\underline{C}}{}^{C} = 2\overline{R}_{sj}.$$

So

$$-\overline{R}_{\underline{i}\underline{j}}{}^{i}{}_{\underline{s}} = \overline{R}_{sj} + \overline{R}_{isj}{}^{i}.$$

From Gauss equation, we also have $R_{sj} = a_{sj} - b_{sj} + \overline{R}^{i}{}_{sij}$. So

$$
\begin{aligned}
\frac{\partial}{\partial t} H_j &= f' a^{ik} h_{ijk} + f'(-a_j^s + R_j^s + \overline{R}_i{}^s{}_j{}^i) H_s \\
&\quad + f' \nabla_j (d^\dagger H) + f' \overline{R}_{sj} H^s + f^{(3)} H_j |H|^2 \\
&\quad + f'' H^i \nabla_i H_j + f'' H^i \nabla_j H_i + f'' H_j d^\dagger H \\
&= f' a^{ik} h_{ijk} - f' b_j^s H_s + f' \nabla_j (d^\dagger H) \\
&\quad + f' \overline{R}_{sj} H^s + f^{(3)} H_j |H|^2 \\
&\quad + f'' H^i \nabla_i H_j + f'' H^i \nabla_j H_i + f'' H_j d^\dagger H \\
&= f' \nabla_j (d^\dagger H) \\
&\quad + f' \overline{R}_{sj} H^s + f^{(3)} H_j |H|^2 \\
&\quad + 2 f'' H^i \nabla_i H_j + f'' H_j d^\dagger H,
\end{aligned}
$$

where we used in the last equality the Kähler-Einstein condition and (2.6) which give

$$
\nabla_i H_j = \nabla_j H_i. \tag{4.14}
$$

So since $(\bar{M}, J, \bar{g})$ is Kähler-Einstein, we get :

$$
\begin{aligned}
\frac{\partial}{\partial t} H_j &= f' \nabla_j (d^\dagger H) + f' \frac{\overline{R}}{2n} H_j + f^{(3)} H_j |H|^2 \\
&\quad + 2 f'' H^i \nabla_i H_j + f'' H_j d^\dagger H \\
&\overset{(4.14)}{=} \nabla_j (f' d^\dagger H) + f' \frac{\overline{R}}{2n} H_j + \nabla_j (f'' |H|^2).
\end{aligned} \tag{4.15}
$$

q.e.d.

Lemma 4.9. *The norm of the mean curvature vector satisfies the evolution equation*

$$
\begin{aligned}
\frac{\partial}{\partial t} |H|^2 &= f' \triangle |H|^2 - 2 f' |\nabla H|^2 + 2 f' b^{ij} H_i H_j - 2 f' \overline{R}^{isj}{}_s H_i H_j \\
&\quad + f' \frac{\overline{R}}{n} |H|^2 + 2 f^{(3)} |H|^4 + 4 f'' H^i H^j \nabla_i H_j + 2 f'' |H|^2 d^\dagger H.
\end{aligned}
$$

Proof: From (4.15), we derive:

$$
\begin{aligned}
\frac{\partial}{\partial t}|H|^2 &= \frac{\partial}{\partial t}(H^j H_j) \\
&\overset{(4.12)}{=} (2f' a_j{}^k H_k + f'\nabla_j(d^\dagger H) + f'\frac{\overline{R}}{2n}H_j + f^{(3)}H_j|H|^2 \\
&\quad +2f'' H^i \nabla_i H_j + f'' H_j d^\dagger H)H^j \\
&\quad +H^j(f'\nabla_j(d^\dagger H) + f'\frac{\overline{R}}{2n}H_j + f^{(3)}H_j|H|^2 \\
&\quad +2f'' H^i \nabla_i H_j + f'' H_j d^\dagger H) \\
&= 2f' a^{jk} H_j H_k + 2f' H^j \nabla_j (d^\dagger H) + f'\frac{\overline{R}}{n}|H|^2 \\
&\quad +2f^{(3)}|H|^4 + 4f'' H^i H^j \nabla_i H_j + 2f''|H|^2 d^\dagger H.
\end{aligned}
$$

We have also

$$
\begin{aligned}
\triangle|H|^2 &= \nabla^k \nabla_k |H|^2 \\
&= \nabla^k \nabla_k (H^j H_j) \\
&= \nabla^k (H_j \nabla_k H^j + H^j \nabla_k H_j) \\
&= 2|\nabla H|^2 + 2H^j \nabla^k \nabla_k H_j \\
&\overset{(4.14)}{=} 2|\nabla H|^2 + 2H^j \nabla^k \nabla_j H_k \\
&\overset{(4.16)}{=} 2|\nabla H|^2 + 2H^j \nabla_j \nabla^k H_k + 2R^k{}_{jk}{}^s H_s H^j \\
&= 2|\nabla H|^2 + 2H^j \nabla_j (d^\dagger H) + 2R^s_j H_s H^j.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial}{\partial t}|H|^2 = f'\triangle|H|^2 - 2f'|\nabla H|^2 + 2f' b^{ij} H_i H_j - 2f'\overline{R}^{isj}{}_s H_i H_j \\
+f'\frac{\overline{R}}{n}|H|^2 + 2f^{(3)}|H|^4 + 4f'' H^i H^j \nabla_i H_j + 2f''|H|^2 d^\dagger H.
\end{aligned}
$$

q.e.d.

Lemma 4.10. $|A|^2$ *satisfies*

$$
\begin{aligned}
\frac{\partial}{\partial t}|A|^2 = f'\triangle|A|^2 - 2f'|\nabla A|^2 + 2f'|b_{is}|^2 + 2f'|A_{iujs} - A_{ijus}|^2 \\
-2f' b^{iu}\overline{R}^s{}_{isu} - 2f' h^{ijk} H^s \overline{R}_{\underline{ijks}} \\
+2f' h^{ijk} \nabla_i \overline{R}^s{}_{js\underline{k}} + 2f' h^{ijk} \nabla^s \overline{R}_{sij\underline{k}} \\
-4f' A^{ij}{}_{su} \overline{R}^s{}_{ij}{}^u + 6f'' a^{ij} \nabla_j H_i + 2f^{(3)} a^{ij} H_i H_j.
\end{aligned}
$$

Proof: First we get

$$\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}(h^{ijk}h_{ijk}) \\
&= h^{ijk}\frac{\partial}{\partial t}h_{ijk} + h_{ijk}\frac{\partial}{\partial t}h^{ijk} \\
&= h^{ijk}\frac{\partial}{\partial t}h_{ijk} + h_{ijk}\frac{\partial}{\partial t}\left(g^{mi}g^{jn}g^{ks}h_{mns}\right) \\
&= (-f'a_i{}^s h_{jks} + f''H_k\nabla_j H_i + f'\nabla_k\nabla_j H_i \\
&\quad + f^{(3)}H_kH_jH_i + f''\nabla_k H_j H_i \\
&\quad + f''\nabla_k H_i H_j - f'a_j^s h_{ksi} - f'H^s\overline{R}_{\underline{i}jk\underline{s}})h^{ijk} \\
&\quad + (2f'a^{mi}h_m^{jk} + 2f'a^{jn}h_n^{ik} + 2f'a^{ks}h_s^{ij})h_{ijk} \\
&\quad + \left(\frac{\partial}{\partial t}h_{ijk}\right)h^{ijk}
\end{aligned}$$

Then, if we take into account the evolution equation for h_{ijk}, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= (-f'a_i{}^s h_{jks} + f''H_k\nabla_j H_i + f'\nabla_k\nabla_j H_i \\
&\quad + f^{(3)}H_kH_jH_i + f''\nabla_k H_j H_i \\
&\quad + f''\nabla_k H_i H_j - f'a_j^s h_{ksi} - f'H^s\overline{R}_{\underline{i}jk\underline{s}})h^{ijk} \\
&\quad + \left(2f'a^{mi}h_m^{jk} + 2f'a^{jn}h_n^{ik} + 2f'a^{ks}h_s^{ij}\right)h_{ijk} \\
&\quad + (-f'a_i{}^s h_{jks} + f''H_k\nabla_j H_i + f'\nabla_k\nabla_j H_i \\
&\quad + f^{(3)}H_kH_jH_i + f''\nabla_k H_j H_i \\
&\quad + f''\nabla_k H_i H_j - f'a_j^s h_{ksi} - f'H^s\overline{R}_{\underline{i}jk\underline{s}})h^{ijk} \\
&= -2f'a^{is}b_{is} + 6f''a^{ij}\nabla_j H_i + 2f'h^{ijk}\nabla_k\nabla_j H_i \\
&\quad + 2f^{(3)}a^{ij}H_iH_j - 2f'a^{js}b_{js} - 2f'h^{ijk}H^s\overline{R}_{\underline{i}jk\underline{s}} \\
&\quad + 2f'a^{is}b_{is} + 2f'a^{js}b_{js} + 2f'a^{ks}b_{ks} \\
&= 2f'a^{is}b_{is} + 6f''a^{ij}\nabla_j H_i + 2f'h^{ijk}\nabla_k\nabla_j H_i \\
&\quad + 2f^{(3)}a^{ij}H_iH_j - 2f'h^{ijk}H^s\overline{R}_{\underline{i}jk\underline{s}}.
\end{aligned}$$

Also

$$\begin{aligned}
\triangle|A|^2 &= \nabla^s\nabla_s(h^{ijk}h_{ijk}) \\
&= \nabla^s(2h^{ijk}\nabla_s h_{ijk}) \\
&= 2|\nabla A|^2 + 2h^{ijk}\nabla^s\nabla_s h_{ijk} \\
&\overset{(2.5)}{=} 2|\nabla A|^2 + 2h^{ijk}\nabla^s\nabla_i h_{sjk} - 2h^{ijk}\nabla^s\overline{R}_{sij\underline{k}}.
\end{aligned}$$

Now setting $h = h_{uvt}dx^u \otimes dx^v \otimes dx^t$, we have

$$
\begin{aligned}
\nabla_s\nabla_m h_{ijk}-\nabla_m\nabla_s h_{ijk} &= \nabla^2 h(\partial_s,\partial_m)(\partial_i,\partial_j,\partial_k)-\nabla^2 h(\partial_m,\partial_s)(\partial_i,\partial_j,\partial_k)\\
&=\{R(\partial_s,\partial_m)h\}(\partial_i,\partial_j,\partial_k)\\
&=\left\{R(\partial_s,\partial_m)(h_{uvt}dx^u\otimes dx^v\otimes dx^t)\right\}(\partial_i,\partial_j,\partial_k)\\
&=h_{uvt}\left\{R(\partial_s,\partial_m)(dx^u\otimes dx^v\otimes dx^t)\right\}(\partial_i,\partial_j,\partial_k)\\
&=h_{uvt}\left\{R(\partial_s,\partial_m)dx^u\otimes dx^v\otimes dx^t\right\}(\partial_i,\partial_j,\partial_k)\\
&\quad +h_{uvt}dx^u\otimes\left\{R(\partial_s,\partial_m)dx^v\otimes dx^t\right\}(\partial_i,\partial_j,\partial_k)\\
&\quad +h_{uvt}dx^u\otimes dx^v\otimes\left\{R(\partial_s,\partial_m)dx^t\right\}(\partial_i,\partial_j,\partial_k)\\
&=h_{uvt}(R_{smw}{}^u dx^w\otimes dx^v\otimes dx^t\\
&\quad +R_{smw}{}^v dx^u\otimes dx^w\otimes dx^t\\
&\quad +R_{smw}{}^t dx^u\otimes dx^v\otimes dx^w)(\partial_i,\partial_j,\partial_k)\\
&=h_{ujk}R_{smi}{}^u+h_{iuk}R_{smj}{}^u+h_{iju}R_{smk}{}^u. \qquad (4.16)
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\triangle|A|^2 &= 2|\nabla A|^2+2h^{ijk}\nabla_i\nabla^s h_{sjk}+2h^{ijk}h_{ujk}R^s{}_{is}{}^u\\
&\quad +2h^{ijk}h_{suk}R^s{}_{ij}{}^u+2h^{ijk}h_{sju}R^s{}_{ik}{}^u-2h^{ijk}\nabla^s\overline{R}_{sij\underline{k}}\\
&\overset{(2.5)}{=} 2|\nabla A|^2+2h^{ijk}\nabla_i\nabla_j H_k\\
&\quad -2h^{ijk}\nabla_i\overline{R}^s{}_{js\underline{k}}+2b^i_u R_i{}^u\\
&\quad +4h^{ijk}h_{suk}R^s{}_{ij}{}^u-2h^{ijk}\nabla^s\overline{R}_{sij\underline{k}}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= f'\triangle|A|^2-2f'|\nabla A|^2+2f'a^{is}b_{is}\\
&\quad +6f''a^{ij}\nabla_j H_i+2f^{(3)}a^{ij}H_iH_j-2f'h^{ijk}H^s\overline{R}_{\underline{ijk}\underline{s}}\\
&\quad +2f'h^{ijk}\nabla_i\overline{R}^s{}_{js\underline{k}}-2f'b_{iu}R^{iu}\\
&\quad -4f'h^{ijk}h_{suk}R^s{}_{ij}{}^u+2f'h^{ijk}\nabla^s\overline{R}_{sij\underline{k}}.
\end{aligned}
$$

Now from Gauss equation (2.4), we have

$$
R_{iu}=a_{iu}-b_{iu}+\overline{R}^s{}_{isu},
$$

so we have

$$
\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= f'\triangle|A|^2 - 2f'|\nabla A|^2 + 2f'|b_{is}|^2 - 2f'b^{iu}\overline{R}^s{}_{isu} \\
&\quad +6f''a^{ij}\nabla_j H_i + 2f^{(3)}a^{ij}H_iH_j - 2f'h^{ijk}H^s\overline{R}_{\underline{i}jk\underline{s}} \\
&\quad +2f'h^{ijk}\nabla_i\overline{R}^s{}_{js\underline{k}} - 4f'A^{ij}{}_{su}R^s{}_{ij}{}^u + 2f'h^{ijk}\nabla^s\overline{R}_{sij\underline{k}} \\
&\overset{(2.4)}{=} f'\triangle|A|^2 - 2f'|\nabla A|^2 + 2f'|b_{is}|^2 - 2f'b^{iu}\overline{R}^s{}_{isu} \\
&\quad +6f''a^{ij}\nabla_j H_i + 2f^{(3)}a^{ij}H_iH_j - 2f'h^{ijk}H^s\overline{R}_{\underline{i}jk\underline{s}} \\
&\quad +2f'h^{ijk}\nabla_i\overline{R}^s{}_{js\underline{k}} - 4f'A^{ij}{}_{su}A^s{}_{ji}{}^u + 4f'A^{ij}{}_{su}A^{su}{}_{ij} \\
&\quad -4f'A^{ij}{}_{su}\overline{R}^s{}_{ij}{}^u + 2f'h^{ijk}\nabla^s\overline{R}_{sij\underline{k}} \\
&= f'\triangle|A|^2 - 2f'|\nabla A|^2 + 2f'|b_{is}|^2 - 2f'b^{iu}\overline{R}^s{}_{isu} \\
&\quad +6f''a^{ij}\nabla_j H_i + 2f^{(3)}a^{ij}H_iH_j - 2f'h^{ijk}H^s\overline{R}_{\underline{i}jk\underline{s}} \\
&\quad +2f'h^{ijk}\nabla_i\overline{R}^s{}_{js\underline{k}} + 2f'|A_{iujs} - A_{ijus}|^2 \\
&\quad -4f'A^{ij}{}_{su}\overline{R}^s{}_{ij}{}^u + 2f'h^{ijk}\nabla^s\overline{R}_{sij\underline{k}}.
\end{aligned}
$$

q.e.d.

Lemma 4.11.

$$
\begin{aligned}
\frac{\partial}{\partial t}d^\dagger H = {} & f'\triangle(d^\dagger H) + 4f'a^{ij}\nabla_i H_j + 2f''H^i\nabla_i(d^\dagger H) \\
& +f''\frac{\overline{R}}{2n}|H|^2 + f'\frac{\overline{R}}{2n}d^\dagger H + f^{(4)}|H|^4 + 2f^{(3)}|H|^2 d^\dagger H \\
& +2f^{(3)}H^i\nabla_i|H|^2 + f''\triangle|H|^2 + f''(d^\dagger H)^2 + 2f''H^lH^ia_{il} \\
& -f''|H|^4 - 2f'H^sH^l\overline{R}^i{}_{si\underline{l}}.
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\frac{\partial}{\partial t}d^\dagger H &= \frac{\partial}{\partial t}(g^{ij}\nabla_i H_j) \\
&\overset{(4.12)}{=} 2f'a^{ij}\nabla_i H_j + g^{ij}\frac{\partial}{\partial t}\nabla_i H_j.
\end{aligned}
$$

Now

$$\nabla_i H_j = \frac{\partial}{\partial x^i}H_j - \Gamma^k_{ij}H_k.$$

So in normal coordinates at a fixed point $P \in L$, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla_i H_j &= \frac{\partial}{\partial x^i}\frac{\partial}{\partial t}H_j - \frac{\partial}{\partial t}\Gamma^k_{ij}H_k \\
&= \nabla_i \frac{\partial}{\partial t}H_j - \frac{\partial}{\partial t}\Gamma^k_{ij}H_k \\
&\overset{(4.15)}{=} \nabla_i(f'\nabla_j(d^\dagger H) + f'\frac{\overline{R}}{2n}H_j + f^{(3)}H_j|H|^2 \\
&\quad +2f''H^k\nabla_k H_j + f''H_j d^\dagger H) \\
&\quad -\frac{1}{2}g^{kl}\left(\nabla_i(\frac{\partial}{\partial t}g_{jl}) + \nabla_j(\frac{\partial}{\partial t}g_{il}) - \nabla_l(\frac{\partial}{\partial t}g_{ij})\right)H_k \\
&= \nabla_i(f'\nabla_j(d^\dagger H) + f'\frac{\overline{R}}{2n}H_j + f^{(3)}H_j|H|^2 \\
&\quad +2f''H^k\nabla_k H_j + f''H_j d^\dagger H) \\
&\quad +H^l(\nabla_i(f'a_{jl}) + \nabla_j(f'a_{il}) - \nabla_l(f'a_{ij})) \\
&= f''H_i\nabla_j(d^\dagger H) + f'\nabla_i\nabla_j(d^\dagger H) + f''\frac{\overline{R}}{2n}H_iH_j + f'\frac{\overline{R}}{2n}\nabla_i H_j \\
&\quad +f^{(4)}|H|^2H_iH_j + f^{(3)}|H|^2\nabla_i H_j \\
&\quad +f^{(3)}H_j\nabla_i|H|^2 + 2f^{(3)}H_iH^k\nabla_k H_j + 2f''\nabla_i H^k\nabla_k H_j \\
&\quad +2f''H^k\nabla_i\nabla_k H_j + f^{(3)}d^\dagger H H_iH_j + f''d^\dagger H\nabla_i H_j \\
&\quad +f''H_j\nabla_i(d^\dagger H) + f''H^lH_ia_{jl} + f''H^lH_ja_{il} - f''|H|^2a_{ij} \\
&\quad +f'H^l\nabla_i a_{jl} + f'H^l\nabla_j a_{il} - f'H^l\nabla_l a_{ij}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}d^\dagger H &\overset{(4.14)}{=} f'\triangle(d^\dagger H) + 2f'a^{ij}\nabla_i H_j + 2f''H^i\nabla_i(d^\dagger H) \\
&\quad +f''\frac{\overline{R}}{2n}|H|^2 + f'\frac{\overline{R}}{2n}d^\dagger H + f^{(4)}|H|^4 + 2f^{(3)}|H|^2 d^\dagger H \\
&\quad +2f^{(3)}H^i\nabla_i|H|^2 + f''\triangle|H|^2 + f''(d^\dagger H)^2 + 2f''H^lH^ia_{il} \\
&\quad -f''|H|^4 + 2f'H^l\nabla^i a_{il} - f'H^l\nabla_l|H|^2 \\
&= f'\triangle(d^\dagger H) + 4f'a^{ij}\nabla_i H_j + 2f''H^i\nabla_i(d^\dagger H) \\
&\quad +f''\frac{\overline{R}}{2n}|H|^2 + f'\frac{\overline{R}}{2n}d^\dagger H + f^{(4)}|H|^4 + 2f^{(3)}|H|^2 d^\dagger H \\
&\quad +2f^{(3)}H^i\nabla_i|H|^2 + f''\triangle|H|^2 + f''(d^\dagger H)^2 + 2f''H^lH^ia_{il} \\
&\quad -f''|H|^4 - 2f'H^sH^l\overline{R}^i{}_{si\underline{l}},
\end{aligned}$$

where the last equality comes from

$$\begin{aligned}
2f'H^l\nabla^i a_{il} &= 2f'H^l\nabla^i(H^s h_{sil}) \\
&= 2f'H^l h_{sil}\nabla^i H^s + 2f'H^lH^s\nabla^i h_{sil} \\
&\overset{(2.5)}{=} 2f'a_{is}\nabla^i H^s + 2f'H^sH^l\nabla_s H_l + 2f'H^sH^l\overline{R}^i{}_{si\underline{l}} \\
&= 2f'a_{is}\nabla^i H^s + f'H^s\nabla_s|H^2| - 2f'H^sH^l\overline{R}^i{}_{si\underline{l}}.
\end{aligned}$$

This completes the proof of the lemma. *q.e.d.*

4.3.1 Evolution equations in the euclidean space

In this section and in the sequel, the ambient Kähler-Einstein manifold is $(\mathbb{R}^{2n}, J, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ is the euclidean metric and $J := i$ is the usual complex structure on $\mathbb{R}^{2n}$. Now the above evolution equations simplify and we obtain:

Lemma 4.12. *For the flow $\frac{d}{dt}F = f'\overrightarrow{H}$ we have the following evolution equations in $\mathbb{R}^{2n}$:*

$$\frac{\partial}{\partial t}g_{ij} = -2f'a_{ij}, \tag{4.17}$$

$$\begin{aligned}\frac{\partial}{\partial t}h_{ijk} = &-f'a_i{}^s h_{jks} + f''H_k\nabla_j H_i + f'\nabla_k\nabla_j H_i\\ &+f^{(3)}H_kH_jH_i + f''\nabla_k H_j H_i\\ &+f''\nabla_k H_i H_j - f'a_j^s h_{ksi},\end{aligned} \tag{4.18}$$

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 = &f'\triangle|A|^2 - 2f'|\nabla A|^2 + 2f'|b_{is}|^2\\ &+2f'|A_{iujs} - A_{ijus}|^2 + 6f''a^{ij}\nabla_j H_i\\ &+2f^{(3)}a^{ij}H_iH_j,\end{aligned} \tag{4.19}$$

$$\frac{\partial}{\partial t}H_j = \nabla_j(f'd^\dagger H) + \nabla_j(f''|H|^2), \tag{4.20}$$

$$\frac{\partial}{\partial t}\alpha = f'\triangle\alpha + f''|\nabla\alpha|^2, \tag{4.21}$$

$$\frac{\partial}{\partial t}d\mu = -f'|H|^2 d\mu, \tag{4.22}$$

$$\begin{aligned}\frac{\partial}{\partial t}|H|^2 = &f'\triangle|H|^2 - 2f'|\nabla H|^2 + 2f'b^{ij}H_iH_j + 2f^{(3)}|H|^4\\ &+4f''H^iH^j\nabla_i H_j + 2f''|H|^2 d^\dagger H,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\frac{\partial}{\partial t}d^\dagger H = &f'\triangle(d^\dagger H) + 4f'a^{ij}\nabla_i H_j + 2f''H^i\nabla_i(d^\dagger H)\\ &+f^{(4)}|H|^4 + 2f^{(3)}|H|^2 d^\dagger H\\ &+2f^{(3)}H^i\nabla_i|H|^2 + f''\triangle|H|^2 + f''(d^\dagger H)^2\\ &+2f''H^lH^i a_{il} - f''|H|^4.\end{aligned} \tag{4.24}$$

Proof: These equations follow from those obtained in the last section for the more general situation. For (4.21) we note that this is a consequence of (4.20) because $H_j = \nabla_j\alpha$. (4.22) is a consequence of (4.17). *q.e.d.*

Now let $\overline{S} \in \mathscr{C}$ be one of the tensors described in Section 4.1. We will also denote by S the $(0,2)$ tensor on L defined by $S_{ij} :=< S(F_i), F_j >$. We get

$$S_{ij} := S_{AB} F_i^A F_j^B ,$$

where $S_{AB} :=< \overline{S}(\partial_A), \partial_B >$.

Lemma 4.13. *S and S^* are symmetric operators, i.e*

$$S_{ij} = S_{ji} \tag{4.25}$$

and

$$S^*_{ij} = S^*_{ji}. \tag{4.26}$$

Proof: From (4.4) and (4.5) we get

$$\overline{S}(\overline{S}^t) = \mathrm{Id}\,.$$

Then, we have

$$\begin{aligned} S_{ij} : &= \left\langle \overline{S}(F_i), F_j \right\rangle \\ &= \left\langle \overline{S}\left(\overline{S}(F_i)\right), \overline{S}(F_j) \right\rangle \\ &\overset{(4.5)}{=} \left\langle F_i, \overline{S}(F_j) \right\rangle \\ &= \left\langle \overline{S}(F_j), F_i \right\rangle \\ &= S_{ji}. \end{aligned}$$

Using the lemma 4.2, we do the same thing for $S^*_{ij} := \left\langle \overline{S}^*(F_i), F_j \right\rangle$ and we will get $S^*_{ij} = S^*_{ji}$. *q.e.d.*

Lemma 4.14. *$F^*\overline{S} := S_{ij} dx^i \otimes dx^j$ holds*

$$\begin{aligned} \frac{\partial}{\partial t} S_{ij} = f' \triangle S_{ij} + f'' H_i H^l S^*_{lj} + f'' H_j H^l S^*_{li} - f' R_i{}^l S_{lj} \\ - f' R_j{}^l S_{li} + 2 f' h_i{}^{mk} h_{jk}{}^n S_{mn}. \end{aligned}$$

Proof: Setting

$$\dot{F}^A := \frac{\partial}{\partial t} F^A$$

we get

$$
\begin{aligned}
\frac{\partial}{\partial t}S_{ij} &= \overline{\nabla}_C S_{AB}\dot{F}^C F_i^A F_j^B + S_{AB}\nabla_i \dot{F}^A F_j^B + S_{AB}\nabla_j \dot{F}^B F_i^A \\
&= S_{AB}\nabla_i (f' H^l \nu_l^A) F_j^B + S_{AB}\nabla_j (f' H^l \nu_l^B) F_i^A \\
&= f'' H_i S_{AB} H^l \nu_l^A F_j^B + f' \nabla_i H^l S_{AB} \nu_l^A F_j^B \\
&\quad + f'' H_j S_{AB} H^l \nu_l^B F_i^A + f' \nabla_j H^l S_{AB} \nu_l^B F_i^A \\
&\quad - f' H^l S_{AB} (h_{il}{}^k F_k^A F_j^B + h_{jl}{}^k F_k^B F_i^A) \\
&= f'' H_i H^l S^*_{lj} + f' \nabla_i H^l S^*_{lj} + f'' H_j H^l S^*_{li} \\
&\quad + f' \nabla_j H^l S^*_{li} - f' a_i{}^k S_{kj} - f' a_j{}^k S_{ik}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\triangle S_{ij} &= \nabla^k \nabla_k (S_{AB} F_i^A F_j^B) \\
&= \nabla^k (h_{ik}{}^l S_{AB} \nu_l^A F_j^B + h_{jk}{}^l S_{AB} \nu_l^B F_i^A) \\
&= \nabla^k h_{ik}{}^l S_{AB} \nu_l^A F_j^B + \nabla^k h_{jk}{}^l S_{AB} \nu_l^B F_i^A \\
&\quad - h_{ik}{}^l S_{AB} (h_l{}^{km} F_m^A F_j^B - h_j{}^{km} \nu_m^B \nu_l^A) \\
&\quad + h_{jk}{}^l S_{AB} (h_i{}^{km} \nu_l^B \nu_m^A - h_l{}^{km} F_m^B F_i^A) \\
&= \nabla_i H^l S^*_{lj} + \nabla_j H^l S^*_{li} - b_i{}^k S_{kj} \\
&\quad - b_j{}^k S_{ki} - 2 h_i{}^{lk} h_{jk}{}^m S_{lm},
\end{aligned}
$$

where we used in the last equation (2.5) and the vanishing of the curvature in $\mathbb{R}^{2n}$ and the lemma 4.13.
We obtain then the result. *q.e.d.*

Defining function $s := g^{ij} S_{ij}$ we get from the lemma 4.14 and from (4.17):

Lemma 4.15. *The function s satisfies the evolution equation*

$$
\frac{\partial}{\partial t} s = f' \triangle s + 4 f' b^{ij} S_{ij} + 2 f'' H^i H^j S^*_{ij}.
$$

4.4 Evolution of graphs

Now in this section, we are interested in Lagrangian graphs in $\mathbb{C}^n$ evolving by functions of its Lagrangian angle. We will consider graphs for which $S := F^*\overline{S} > \epsilon\langle\cdot,\cdot\rangle$ for some $\epsilon \in (0,1)$ and for some tensor $\overline{S}$ as in Section 4.1. Since $\overline{S}^2 = Id$, then the eigenvalues of $\overline{S}$ are either 1 or -1. If X is a eigenvector of $\overline{S}$ such that $\overline{S}(X) = X$, then JX is also eigenvector of $\overline{S}$ because by the condition (4.3) we have $\overline{S}(JX) = -J\overline{S}(X) = -JX$. So, 1 and -1 are exactly the two eigenvalues of $\overline{S}$ and J maps the eigenspace belonging to 1 to that belonging to -1. Consequently we can split $\mathbb{R}^{2n}$ into the orthogonal direct sum

$$\mathbb{R}^{2n} = E \oplus D,$$

where E is the linear hull of all eigenvectors e belonging to 1 and $D := JE$ is the linear hull of eigenvectors d belonging to -1.
Now let $L \subset \mathbb{R}^{2n}$ be a compact Lagrangian submanifold in $\mathbb{R}^{2n}$ such that its universal cover $\widetilde{L}$ is $\mathbb{R}^n$ immersed as a Lagrangian graph into $\mathbb{R}^{2n}$. Let

$$F : \mathbb{R}^n \to \mathbb{R}^{2n}$$

$$F(x) := x^i e_i + \delta^{ij} u_i d_j$$

be this Lagrangian immersion and where $u_1, \ldots, u_n : \mathbb{R}^n \to \mathbb{R}$ are functions and $e_1, \ldots, e_n$ is a orthonormal frame spanning E and $d_j := Je_j$.

The tangent vectors $F_i := \frac{\partial F}{\partial x^i}$ are given by

$$F_i = e_i + \delta^{kl} u_{ki} d_l \tag{4.27}$$

where $u_{ki} := \frac{\partial u_i}{\partial x^k}$.
The Lagrangian condition $\langle JF_i, F_j\rangle = 0$ implies that $u_{ij} - u_{ji} = 0$. This means that the 1-form on $\mathbb{R}^n$, $\beta := u_i dx^i$ is closed. Since $H^1(\mathbb{R}^n) = 0$ then β must be exact. So there exist a function $u : \mathbb{R}^n \to \mathbb{R}$ such that $\beta = du$ which means $u_i = \frac{\partial u}{\partial x^i}$ for any $i = 1, \ldots, n$ and on all $\mathbb{R}^n$.
Here the Lagrangian angle α exists for every time as long as the solution of (4.1) exists and they are given by either

$$\alpha = -arctan(\frac{a}{b}),$$

or by

$$\alpha = arctan(\frac{b}{a})$$

depending on whether

$$a := Im(det(\delta_{kl} + iu_{kl}))$$

or

$$b := Re(det(\delta_{kl} + iu_{kl}))$$

is nonzero.
Indeed a and b cannot both be zero because the induced metric

$$g_{ij} := \langle F_i, F_j\rangle = \delta_{ij} + \delta^{kl} u_{ik} u_{jl} = I + A^2 = (I + iA)(I - iA)$$

where I the identity matrix and $A = (u_{ij})_{i,j}$.
So $det(g_{ij}) = det(I+ia)det(I-iA) = det(I+iA)det(\overline{I+iA}) = (b+ia)(\overline{b+ia})$.
Then

$$det(g_{ij}) = a^2 + b^2$$

and must not vanish.
As in [43] we transform the flow (4.1) into a parabolic equation for u, because

$$\frac{d}{dt}F = J\nabla f = f' H^m \nu_m = f' H^m (d_m - \delta^{lp} u_{lm} e_p)$$

implies the equations

$$\frac{dx^i}{dt} = -f' H^m \delta^{li} u_{lm} \tag{4.28}$$

and

$$\delta^{ip}\frac{du_p}{dt} = f' H^i \tag{4.29}$$

But

$$\frac{du_p}{dt} = \frac{\partial u_p}{\partial t} + u_{pl}\frac{dx^l}{dt}$$

and since $H^m = \nabla^m \alpha$, then (4.29) and (4.28) implies

$$\delta^{jp}\frac{\partial^2 u}{\partial t \partial x^p} - f' \delta^{jp} u_{pl} \nabla^m \alpha \delta^{li} u_{im} = f' \nabla^j \alpha.$$

Now

$$\begin{aligned} g_{ij} &:= \langle F_i, F_j \rangle \\ &= \delta_{ij} + \delta^{kl} u_{ik} u_{jl} \end{aligned} \tag{4.30}$$

so

$$\delta^{jp}\frac{\partial^2 u}{\partial t \partial x^p} - f' \delta^{jp}(g_{pm} - \delta_{pm})\nabla^m \alpha = f' \nabla^j \alpha,$$

or

$$\frac{\partial^2 u}{\partial t \partial x^j} - f' \delta^{jp} \nabla_p \alpha + f' \nabla^j \alpha = f' \nabla^j \alpha,$$

or

$$\frac{\partial^2 u}{\partial t \partial x^j} - f' \nabla_j \alpha = 0.$$

Then we obtain the following parabolic equation for u :

$$P[u] := \frac{\partial u}{\partial t} - f = 0 \tag{4.31}$$

with $d\alpha = H$. We now compute

$$\frac{\partial P[u]}{\partial u_{ij}} = \frac{\partial f}{\partial u_{ij}}.$$

Now $H_k = g^{ij}h_{ijk} = g^{ij}u_{ijk}$ and

$$f'H_k = \frac{\partial}{\partial x^k}f = \frac{\partial f}{\partial u_{ij}}\frac{\partial u_{ij}}{\partial x^k} = \frac{\partial f}{\partial u_{ij}}u_{ijk}$$

So $\frac{\partial}{\partial u_{ij}}f = f'g^{ij}$ then

$$\frac{\partial P[u]}{\partial u_{ij}} = f'g^{ij}$$

which means that $P[u]$ is always parabolic. We also have

$$\begin{aligned}
\frac{\partial^2 P[u]}{\partial u_{ij}\partial u_{kl}} &= \frac{\partial f' g^{ij}}{\partial u_{kl}} \\
&= f''g^{kl}g^{ij} - f'g^{is}g^{jm}\frac{\partial g_{sm}}{\partial u_{kl}} \\
&\overset{(4.30)}{=} f''g^{kl}g^{ij} - f'g^{is}g^{jm}(\delta^{pq}u_{ps}\delta^k_q\delta^l_m + \delta^{pq}u_{qm}\delta^k_p\delta^l_s) \\
&= f''g^{kl}g^{ij} - f'(g^{is}g^{jl} + g^{il}g^{js})\delta^{pk}u_{ps}
\end{aligned}$$

where the second equality comes from $\frac{\partial f'}{\partial u_{kl}} = f''\frac{\partial \alpha}{\partial u_{kl}}$ and $\frac{\partial \alpha}{\partial u_{kl}} = g^{kl}$ because $\frac{\partial \alpha}{\partial x^i} = H_i = g^{kl}u_{kli}$ and $\frac{\partial \alpha}{\partial x^i} = \frac{\partial \alpha}{\partial u_{kl}}\frac{\partial u_{kl}}{\partial x^i} = \frac{\partial \alpha}{\partial u_{kl}}u_{kli}$.
Setting $c^{ij} := v^{im}v_{jm}$, then for any symmetric tensor v_{ij} we have

$$\begin{aligned}
\frac{\partial^2 P[u]}{\partial u_{ij}\partial u_{kl}}v_{ij}v_{kl} &= c^{ij}(f''g_{ij} - 2f'u_{ij}) \\
&= c^{ij}(f''g_{ij} + f'S^*_{ij}) \\
&= f'c^{ij}(S^*_{ij} + \frac{f''}{f'}g_{ij}),
\end{aligned}$$

where the second equality comes from $S^*_{ij} = -2u_{ij}$. Indeed we have :

$$\overline{S}^*_{ij} : = \overline{S}(JF_i, F_j).$$

Since

$$\overline{S}(e_i, e_j) = \delta_{ij},$$

$$\overline{S}(d_i, d_j) = -\delta_{ij},$$
$$\overline{S}(e_i, d_j) = \overline{S}(d_i, e_j) = 0,$$

then from (4.27), we get

$$\begin{aligned} S_{ij} &:= \overline{S}(F_i, F_j) \\ &= \delta_{ij} - \delta^{kl} u_{ik} u_{jl} \end{aligned} \tag{4.32}$$

and

$$\overline{S}^*_{ij} = -2u_{ij}.$$

Now the operator $P[u]$ is called concave if

$$\frac{\partial^2 P[u]}{\partial u_{ij} \partial u_{kl}} v_{ij} v_{kl} \leq 0.$$

Lemma 4.16. *The operator $P[u] = \frac{\partial u}{\partial t} - f$ is concave.*

Proof: Since the tensor c^{ij} is positive defined, so we only have to prove that the tensor $M_{ij} := S^*_{ij} + \frac{f''}{f'} g_{ij}$ holds $M_{ij} \leq 0$. From (4.7), we have

$$M_{ij} \leq (1 + \frac{f''}{f'}) g_{ij}.$$

So with $\tilde{f} \in \tilde{\mathcal{F}}_{a,b,\epsilon}$, $M_{ij} \leq 0$ and we get the result. *q.e.d.*

Lemma 4.17. *Assume there exist $\epsilon > 0$ and $\overline{S} \in \mathscr{C}$ such that*

$$M_{ij} := S_{ij} - \epsilon g_{ij} > 0$$

holds on a compact L at $t = 0$. Then this is also true for $t \in [0, T)$.

Proof: From evolution equation of S_{ij} and of the metric, we obtain :

$$\begin{aligned} \frac{\partial}{\partial t} M_{ij} = {} & f' \triangle M_{ij} + f' b^n_i S_{nj} + 2\epsilon f' a_{ij} \\ & + f' b^n_j S_{ni} - f' a^n_i S_{nj} - f' a^n_j S_{ni} \\ & + 2 f' {h_i}^{mk} {h_{jk}}^n S_{mn} + f'' H_i H^k S^*_{kj} + f'' H_j H^k S^*_{ki}. \end{aligned}$$

Setting

$$\begin{aligned} N_{ij} := {} & f' b^n_i S_{nj} + 2\epsilon f' a_{ij} \\ & + f' b^n_j S_{ni} - f' a^n_i S_{nj} - f' a^n_j S_{ni} \\ & + 2 f' {h_i}^{mk} {h_{jk}}^n S_{mn} + f'' H_i H^k S^*_{kj} + f'' H_j H^k S^*_{ki}. \end{aligned}$$

To get the proof, we need to use the theorem 3.1. So we must show that $N_{ij}V^iV^j \geq 0$ for any null eigenvector V of $M_{ij} := S_{ij} - \epsilon g_{ij}$, that occurs for the first time t_0 at the point P.
Since $M_{ij}V^i = 0$ then $S_{ij}V^i = \epsilon V_j$. So, we get :

$$N_{ij}V^iV^j = 2\epsilon f' b_{ij}V^iV^j + 2f' {h_i}^{mk}{h_{jk}}^n S_{mn}V^iV^j + 2f'' H_i H^k S^*_{kj}V^iV^j.$$

But
$b_{ij}V^iV^j := h_{ikl}{h_j}^{kl}V^iV^j = |V^ih_{ijk}|^2 \geq \frac{1}{n}(trace(V^ih_{ijk}))^2 = \frac{1}{n}(\langle H,V\rangle)^2$.
Now, by lemma 4.13, S and S^* are symmetric operators, then we can diagonalize them. Since by (4.6), S and S^* commute, we can choose orthonormal vectors $(e_1, ..., e_n)$ for $T_pL_{t_0}$ such that S_{ij} and S^*_{ij} become diagonal. We also choose $e_1 = V$ then $V^i = \delta^1_i$. We will denote $\lambda_i, i = 1, \cdots, n$ the eigenvalues of S_{ij} and $\lambda^*_i, i = 1, \cdots, n$ the one of S^*_{ij}. So, choosing, λ_1 the smallest one, we get

$$\begin{aligned} 2f'{h_i}^{mk}{h_{jk}}^n S_{mn}V^iV^j &= 2f'h_1^{mk}h_{k1}^n S_{mn} = 2f'h_1^{mk}h_{k1}^m\lambda_m = 2f'|h_1^{mk}|^2\lambda_m \\ &\geq 2f'|h_1^{mk}|^2\lambda_1 \geq \frac{2f'}{n}(H_1)^2\lambda_1 = \frac{2\epsilon f'}{n}(\langle H,V\rangle)^2. \end{aligned}$$

So

$$N_{ij}V^iV^j \geq \frac{4\epsilon}{n}f'(< H, V >)^2 + 2f''H_iH^kS^*_{kj}V^iV^j.$$

Now $2f''H_iH^kS^*_{kj}V^iV^j = 2f''\lambda^*_1(H_1)^2(V^1)^2 = 2f''\lambda^*_1(\langle H,V\rangle)^2$. So

$$N_{ij}V^iV^j \geq (\frac{4\epsilon}{n}f' + 2f''\lambda^*_1)(\langle H,V\rangle)^2.$$

But by lemma 4.3, we have $(\lambda_1)^2 + (\lambda^*_1)^2 = 1$, then $f''\lambda^*_1 \geq -|f''|\sqrt{1-\lambda_1^2} = -|f''|\sqrt{1-\epsilon^2}$. So

$$N_{ij}V^iV^j \geq (\frac{4\epsilon}{n}f' - 2|f''|\sqrt{1-\epsilon^2})(\langle H,V\rangle)^2.$$

So for $f := \tilde{f} \circ \alpha$ with $\tilde{f} \in \mathcal{F}_{a,b,\epsilon}$ we have $\frac{4\epsilon}{n}f' - 2|f''|\sqrt{1-\epsilon^2} \geq 0$ and then $N_{ij}V^iV^j \geq 0$. *q.e.d.*

Then

Lemma 4.18. *Assume that* $t = 0$ *we have* $F_0^*\overline{S}(V,V) > \epsilon|V|^2\ \forall V \in TL, V \neq 0$. *Then we have uniform* C^2*-estimates in space directions for (4.31).*

Proof: Since $S_{ij} := S(F_i, F_j) = \delta_{ij} - \delta^{kl}u_{ik}u_{lj}$ (see (4.32)), then from lemma 4.17, we get the result. *q.e.d.*

Consequently, we have

Lemma 4.19. *Assume that* $t = 0$ *we have* $F_0^* \overline{S}(V, V) > \epsilon |V|^2 \ \forall V \in TL, V \neq 0$. *Then the metrics are uniformly equivalent.*

Proof: $g_{ij} := \delta_{ij} + \delta^{kl} u_{ik} u_{lj}$ and from lemma 4.18, we get the result. *q.e.d.*

Lemma 4.20. *Let* $\tilde{g}_{ij}$ *be the following, to* g_{ij} *conformally equivalent metric*

$$\tilde{g}_{ij} := \frac{1}{f'} g_{ij} \,.$$

If r *is the real function that satisfies* $r' = (f')^{\frac{n}{2}}$, *then* $r \circ \alpha$ *(again denoted by* r*) satisfies the evolution equation*

$$\dot{r} = \tilde{\Delta} r \,,$$

where $\tilde{\Delta}$ *is the Laplace-Beltrami operator w.r.t. the metric* $\tilde{g}$.

Proof: For a general smooth function h we have

$$\tilde{\Delta} h = f' \Delta h - \frac{n-2}{2} f'' \langle \nabla \alpha, \nabla h \rangle \,, \tag{4.33}$$

where ∇ denotes the gradient w.r.t. g. To see this we compute the Christoffel symbols of $\tilde{g}_{ij}$

$$\begin{aligned} \tilde{\Gamma}^k_{ij} &= \frac{1}{2} \tilde{g}^{kl} (\tilde{g}_{il,j} + \tilde{g}_{jl,i} - \tilde{g}_{ij,l}) \\ &= \Gamma^k_{ij} - \frac{f''}{2f'} g^{kl} (g_{lj} \nabla_i \alpha + g_{li} \nabla_j \alpha - g_{ij} \nabla_l \alpha) \\ &= \Gamma^k_{ij} - \frac{f''}{2f'} H_i \delta^k_j - \frac{f''}{2f'} H_j \delta^k_i + \frac{f''}{2f'} g_{ij} H^k \end{aligned} \tag{4.34}$$

so that

$$\begin{aligned} \tilde{\Delta} h &= \tilde{g}^{ij} (h_{ij} - \tilde{\Gamma}^k_{ij} h_k) \\ &= f' g^{ij} (h_{ij} - \Gamma^k_{ij} h_k + \frac{f''}{2f'} g^{kl} (g_{lj} \nabla_i \alpha + g_{li} \nabla_j \alpha - g_{ij} \nabla_l \alpha) h_k) \\ &= f' \Delta h - \frac{n-2}{2} f'' \langle \nabla \alpha, \nabla h \rangle \,. \end{aligned}$$

Then we compute

$$\begin{aligned} \dot{r} &= r' \dot{\alpha} \\ &= r' (f' \Delta \alpha + f'' |\nabla \alpha|^2) \\ &= f' \Delta r + (r' f'' - f' r'') |\nabla \alpha|^2 \\ &\overset{(4.33)}{=} \tilde{\Delta} r + \frac{n-2}{2} f'' \langle \nabla \alpha, \nabla r \rangle + (r' f'' - f' r'') |\nabla \alpha|^2 \\ &= \tilde{\Delta} r + (\frac{n}{2} r' f'' - f' r'') |\nabla \alpha|^2 \\ &= \tilde{\Delta} r \,, \end{aligned}$$

since $\frac{n}{2}r'f'' - f'r'' = 0$. *q.e.d.*

Now, let r be a solution of this differential equation :

$$r' = (f')^{\frac{n}{2}}$$
$$r(0) = 0.$$

We can see that r must be positive because it is increasing function and $r(0) = 0$. And by lemma 4.20, r is a positive solution of a heat equation. We now have the following Harnack inequality for a positive solution of heat equation, proved by Huai-Dong Cao in [5] :

Theorem 4.3 (Huai-Dong Cao) *Let M be a compact manifold of dimension n and let $g_{ij}(t)$, $0 \leq t < \infty$, be a family of Riemannian metrics on M with the following properties :*

$$C_1 g_{ij}(0) \leq g_{ij}(t) \leq C_2 g_{ij}(0), \tag{4.35}$$

$$|\frac{\partial g_{ij}}{\partial t}|(t) \leq C_3 g_{ij}(0), \tag{4.36}$$

$$R_{ij}(t) \geq -K g_{ij}(0), \tag{4.37}$$

where C_1, C_2, C_3, and K are positive constants independent of t. Let $\triangle_t$ denote the Laplace operator of the metric $g_{ij}(t)$. If $\phi(x,t)$ is a positive solution for the equation

$$\frac{\partial}{\partial t}\phi(x,t) = \triangle_t \phi(x,t)$$

on $M \times [0,\infty)$, then for any $\alpha > 1$, we have

$$Sup_{x\in M}\phi(x,t_1) \leq inf_{x\in M}\phi(x,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{n}{2}} \exp(\frac{1}{4(t_2-t_1)}C_2^2 d^2 + (\frac{n\alpha K}{2(\alpha-1)} + C_2C_3(n+A))(t_2-t_1))$$

where d is the diameter of M measured by the metric $g_{ij}(0)$, $A = sup\|\nabla^2 log\phi\|$ and $0 < t_1 < t_2 < \infty$.

Now we want to apply this Harnack inequality theorem of Cao to r, a positive solution of heat equation, such that

$$r' = (f')^{\frac{n}{2}}$$
$$r(0) = 0$$

(see lemma 4.20.) Here the metric is $\tilde{g}_{ij} := \frac{1}{f'}g_{ij}$. From 4.21, α is uniformly bounded and since $f' := f'(\alpha)$ is smooth, then $\frac{1}{f'}$ is uniformly bounded too.

Now by lemma 4.19, the metric $\tilde{g}_{ij}(t)$ is uniformly equivalent. So the first assumption of the theorem, i.e (4.35), is satisfied. To check the second assumption, we have to compute $\frac{\partial}{\partial t}\tilde{g}_{ij}$. We have

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}_{ij} &= \frac{\partial}{\partial t}(\frac{1}{f'}g_{ij}) \\ &= -\frac{f''}{(f')^2}\frac{\partial\alpha}{\partial t}g_{ij} + \frac{1}{f'}\frac{\partial}{\partial t}g_{ij} \\ &\overset{(4.21)}{=} -\frac{f''}{(f')^2}(f'\triangle\alpha + f''|\nabla\alpha|^2)g_{ij} + \frac{1}{f'}(-2f'a_{ij}) \\ &= -\frac{f''}{(f')^2}(f' trace(\nabla H) + f''|H|^2)g_{ij} - 2a_{ij}.\end{aligned}$$

We now need to prove that the second form fundamental $(h_{ij})_{i,j}$ is bounded and all its covariant derivatives. To get this, we utilize the $C^{2,\alpha}$-estimate in space and the $C^{1,\alpha}$-estimate in time for nonlinear parabolic equations (4.31) by Krylov [30] or [31] (see Section 5.5 in this paper) to get C^∞-estimates for u. To apply it, we need the uniform C^2-estimate in space for u and the uniform C^1-estimate in time for u and the concavity of the operator (4.31).

So, assume that $t = 0$ we have $F_0^*\overline{S}(V,V) > \epsilon|V|^2\ \forall V \in TL, V \neq 0$. Then lemma 4.18 gives the uniform C^2-estimate in space for u. We also have C^1-estimates in time for u because by maximum principle α is uniformly bounded. Now by lemma 4.16, the operator $P[u]$ is concave. So with $C^{2,\alpha}$-estimate in space and the $C^{1,\alpha}$-estimate in time for nonlinear parabolic equations (4.31), standard Schauder estimates give C^∞-estimates both in space and time for u. In particular the full norm of the second fundamental form is uniformly bounded because $h_{ijk} = u_{ijk}$ and all its covariant derivatives are bounded as well. This implies that H, ∇H and a_{ij} are uniformly bounded. Indeed, from

$$|h_{ijk} - c(H_i g_{jk}) + H_j g_{ki} + H_k g_{ij}|^2 \geq 0$$

we obtain, choosing $c = \frac{1}{n+2}$, the inequality

$$|H|^2 \leq \frac{n+2}{3}|A|^2.$$

So this gives a uniform bound for H. We also know that $|\nabla H|^2 \leq c(n)|\nabla A|^2$ where $c(n)$ is a constant dependent only of $n = \dim L$. And this gives a uniform bound of $|\nabla H|$. Now about a_{ij}, we have $|a_{ij}|^2 = b_{ij}H^iH^j = \sum_{i=1}^n \beta_i(H^i)^2$ where we write $b_{ij} = diag(\beta_1, \beta_2, \cdots, \beta_n)$ in a base of eigenvectors ($(b_{ij})_{i,j}$ is indeed symmetric matrix then we can diagonalize it). Now $(b_{ij})_{i,j}$ is positive defined because for every vector field V, we have $b_{ij}V^iV^j := h_{ikl}h_j{}^{kl}V^iV^j = |V^ih_{ijk}|^2 \geq 0$. That means all β_i are positive. So $|a_{ij}|^2 \leq \sum_{i=1}^n \beta_i|H|^2 = trace((b_{ij}))|H|^2 = |A|^2|H|^2$ and then $|a_{ij}|$ is also uniformly bounded. Then, we get a uniform bound for $\frac{\partial}{\partial t}\tilde{g}_{ij}$. Since the metrics $\tilde{g}_{ij}$ are uniformly bounded,

the second assumption, i.e (4.36), is also satisfied.
To get the last assumption (4.37), we need to compute $\tilde{R}_{ij}$ where $\tilde{R}_{ijkl}$ is the curvature operator of the metric $\tilde{g}_{ij}$. We have

$$
\begin{aligned}
\tilde{R}(\partial_i,\partial_j)\partial_k &= \tilde{\nabla}_{\partial_i}\tilde{\nabla}_{\partial_j}\partial_k - \tilde{\nabla}_{\partial_j}\tilde{\nabla}_{\partial_i}\partial_k \\
&= \tilde{\nabla}_{\partial_i}(\tilde{\Gamma}^s_{jk}\partial_s) - \tilde{\nabla}_{\partial_j}(\tilde{\Gamma}^s_{ik}\partial_s) \\
&= \partial_i(\tilde{\Gamma}^s_{jk})\partial_s + \tilde{\Gamma}^s_{jk}\tilde{\nabla}_{\partial_i}\partial_s - \partial_j(\tilde{\Gamma}^s_{ik})\partial_s - \tilde{\Gamma}^s_{ik}\tilde{\nabla}_{\partial_j}\partial_s \\
&= \partial_i(\tilde{\Gamma}^s_{jk})\partial_s - \partial_j(\tilde{\Gamma}^s_{ik})\partial_s + \tilde{\Gamma}^s_{jk}\tilde{\Gamma}^m_{is}\partial_m - \tilde{\Gamma}^s_{ik}\tilde{\Gamma}^m_{js}\partial_m \qquad (4.38) \\
&\overset{(4.34)}{=} \partial_i(\Gamma^s_{jk})\partial_s - \partial_j(\Gamma^s_{ik})\partial_s - \partial_i\left(\frac{f''}{2f'}H_j\right)\partial_k \\
&\quad -\partial_i\left(\frac{f''}{2f'}H_k\right)\partial_j + \partial_i\left(\frac{f''}{2f'}g_{jk}H^s\right)\partial_s \\
&\quad +\partial_j\left(\frac{f''}{2f'}H_i\right)\partial_k + \partial_j\left(\frac{f''}{2f'}H_k\right)\partial_i - \partial_j\left(\frac{f''}{2f'}g_{ik}H^s\right)\partial_s \\
&\quad +\left(\Gamma^s_{jk} - \frac{f''}{2f'}H_j\delta^s_k - \frac{f''}{2f'}H_k\delta^s_j + \frac{f''}{2f'}g_{jk}H^s\right)(\Gamma^m_{is} - \frac{f''}{2f'}H_i\delta^m_s \\
&\quad -\frac{f''}{2f'}H_s\delta^m_i + \frac{f''}{2f'}g_{is}H^m)\partial_m \\
&\quad -\left(\Gamma^s_{ik} - \frac{f''}{2f'}H_i\delta^s_k - \frac{f''}{2f'}H_k\delta^s_i + \frac{f''}{2f'}g_{ik}H^s\right)(\Gamma^m_{js} - \frac{f''}{2f'}H_j\delta^m_s \\
&\quad -\frac{f''}{2f'}H_s\delta^m_j + \frac{f''}{2f'}g_{js}H^m)\partial_m.
\end{aligned}
$$

So,

$$
\begin{aligned}
\tilde{R}(\partial_i,\partial_j)\partial_k = {} & \partial_i(\Gamma^s_{jk})\partial_s - \partial_j(\Gamma^s_{ik})\partial_s + \Gamma^s_{jk}\Gamma^m_{is}\partial_m - \Gamma^s_{ik}\Gamma^m_{js}\partial_m \\
& -\nabla_i\left(\frac{f''}{2f'}H_j\right)\partial_k - \frac{f''}{2f'}\Gamma^s_{ij}H_s\partial_k - \nabla_i\left(\frac{f''}{2f'}H_k\right)\partial_j \\
& -\frac{f''}{2f'}\Gamma^s_{ik}H_s\partial_j + g_{jk}\nabla_i\left(\frac{f''}{2f'}H^s\right)\partial_s + \frac{f''}{2f'}\Gamma^m_{ij}g_{mk}H^s\partial_s \\
& +\frac{f''}{2f'}\Gamma^m_{ik}g_{jm}H^s\partial_s - \frac{f''}{2f'}\Gamma^s_{im}g_{jk}H^m\partial_s + \nabla_j\left(\frac{f''}{2f'}H_i\right)\partial_k \\
& +\frac{f''}{2f'}\Gamma^s_{ij}H_s\partial_k + \nabla_j\left(\frac{f''}{2f'}H_k\right)\partial_i + \frac{f''}{2f'}\Gamma^s_{jk}H_s\partial_i \\
& -g_{ik}\nabla_j\left(\frac{f''}{2f'}H^s\right)\partial_s - \frac{f''}{2f'}\Gamma^m_{ij}g_{mk}H^s\partial_s - \frac{f''}{2f'}\Gamma^m_{jk}g_{im}H^s\partial_s \\
& +\frac{f''}{2f'}\Gamma^s_{jm}g_{ik}H^m\partial_s - \frac{f''}{2f'}\Gamma^s_{jk}H_i\partial_s - \frac{f''}{2f'}\Gamma^s_{jk}H_s\partial_i \\
& +\frac{f''}{2f'}\Gamma^s_{jk}g_{is}H^m\partial_m - \frac{f''}{2f'}\Gamma^m_{ik}H_j\partial_m + \frac{(f'')^2}{4(f')^2}H_iH_j\partial_k \\
& +\frac{(f'')^2}{4(f')^2}H_jH_k\partial_i - \frac{(f'')^2}{4(f')^2}g_{ik}H_jH^m\partial_m - \frac{f''}{2f'}\Gamma^m_{ij}H_k\partial_m \\
& +\frac{(f'')^2}{4(f')^2}H_iH_k\partial_j + \frac{(f'')^2}{4(f')^2}H_jH_k\partial_i - \frac{(f'')^2}{4(f')^2}g_{ij}H_kH^m\partial_m \\
& +\frac{f''}{2f'}\Gamma^m_{is}g_{jk}H^s\partial_m - \frac{(f'')^2}{4(f')^2}g_{jk}H_iH^m\partial_m - \frac{(f'')^2}{4(f')^2}g_{jk}|H|^2\partial_i \\
& +\frac{(f'')^2}{4(f')^2}g_{jk}H_iH^m\partial_m + \frac{f''}{2f'}\Gamma^s_{ik}H_j\partial_s + \frac{f''}{2f'}\Gamma^s_{ik}H_s\partial_j \\
& -\frac{f''}{2f'}\Gamma^s_{ik}g_{js}H^m\partial_m + \frac{f''}{2f'}\Gamma^m_{jk}H_i\partial_m - \frac{(f'')^2}{4(f')^2}H_iH_j\partial_k \\
& -\frac{(f'')^2}{4(f')^2}H_iH_k\partial_j + \frac{(f'')^2}{4(f')^2}g_{jk}H_iH^m\partial_m + \frac{f''}{2f'}\Gamma^m_{ij}H_k\partial_m \\
& -\frac{(f'')^2}{4(f')^2}H_jH_k\partial_i - \frac{(f'')^2}{4(f')^2}H_iH_k\partial_j + \frac{(f'')^2}{4(f')^2}g_{ij}H_kH^m\partial_m \\
& -\frac{f''}{2f'}\Gamma^m_{js}g_{ik}H^s\partial_m + \frac{(f'')^2}{4(f')^2}g_{ik}H_jH^m\partial_m + \frac{(f'')^2}{4(f')^2}g_{ik}|H|^2\partial_j \\
& -\frac{(f'')^2}{4(f')^2}g_{ik}H_jH^m\partial_m.
\end{aligned}
$$

But $\partial_i(\Gamma^s_{jk})\partial_s - \partial_j(\Gamma^s_{ik})\partial_s + \Gamma^s_{jk}\Gamma^m_{is}\partial_m - \Gamma^s_{ik}\Gamma^m_{js}\partial_m = R(\partial_i,\partial_j)\partial_k$ (compare with the proof of (4.38).)

$$\begin{aligned}
\tilde{R}(\partial_i,\partial_j)\partial_k &= R(\partial_i,\partial_j)\partial_k - \nabla_i\left(\frac{f''}{2f'}H_j\right)\partial_k - \nabla_i\left(\frac{f''}{2f'}H_k\right)\partial_j \\
&+g_{jk}\nabla_i\left(\frac{f''}{2f'}H^s\right)\partial_s + \nabla_j\left(\frac{f''}{2f'}H_i\right)\partial_k + \nabla_j\left(\frac{f''}{2f'}H_k\right)\partial_i \\
&-g_{ik}\nabla_j\left(\frac{f''}{2f'}H^s\right)\partial_s + \frac{(f'')^2}{4(f')^2}H_jH_k\partial_i - \frac{(f'')^2}{4(f')^2}H_iH_k\partial_j \\
&+\frac{(f'')^2}{4(f')^2}g_{jk}H_iH^m\partial_m - \frac{(f'')^2}{4(f')^2}g_{ik}H_jH^m\partial_m \\
&+\frac{(f'')^2}{4(f')^2}g_{ik}|H|^2\partial_j - \frac{(f'')^2}{4(f')^2}g_{jk}|H|^2\partial_i.
\end{aligned}$$

Now,

$$\begin{aligned}
\tilde{R}_{jk} :&= \tilde{g}^{il}\tilde{R}_{ijlk} \\
&= f'g^{il}\tilde{R}_{ijlk} \\
&= f'g^{il}\tilde{g}\left(\tilde{R}(\partial_i,\partial_j)\partial_k,\partial_l\right) \\
&= f'g^{il}\frac{1}{f'}g\left(\tilde{R}(\partial_i,\partial_j)\partial_k,\partial_l\right) \\
&= g^{il}g\left(\tilde{R}(\partial_i,\partial_j)\partial_k,\partial_l\right).
\end{aligned}$$

So, we get

$$\begin{aligned}
\tilde{R}_{jk} &= R_{jk} - \nabla_k\left(\frac{f''}{2f'}H_j\right) - \nabla_j\left(\frac{f''}{2f'}H_k\right) \\
&+g_{jk}\nabla^i\left(\frac{f''}{2f'}H_i\right) + \nabla_j\left(\frac{f''}{2f'}H_k\right) + n\nabla_j\left(\frac{f''}{2f'}H_k\right) \\
&-\nabla_j\left(\frac{f''}{2f'}H_k\right) + n\frac{(f'')^2}{4(f')^2}H_jH_k - \frac{(f'')^2}{4(f')^2}H_jH_k \\
&+\frac{(f'')^2}{4(f')^2}g_{jk}|H|^2 - \frac{(f'')^2}{4(f')^2}H_jH_k + \frac{(f'')^2}{4(f')^2}g_{jk}|H|^2 \\
&-n\frac{(f'')^2}{4(f')^2}g_{jk}|H|^2 \\
&= R_{jk} + (n-2)\nabla_j\left(\frac{f''}{2f'}H_k\right) + (n-2)\frac{(f'')^2}{4(f')^2}H_jH_k \\
&+\left(\nabla^i\left(\frac{f''}{2f'}H_i\right) - (n-2)\frac{(f'')^2}{4(f')^2}|H|^2\right)g_{jk}. \qquad (4.39)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\tilde{R}_{jk} &= R_{jk} + (n-2)\nabla_j\nabla_k\ln(f')^{\frac{1}{2}} + (n-2)\frac{(f'')^2}{4(f')^2}H_jH_k \\
&\qquad + \left(\triangle\ln(f')^{\frac{1}{2}} - (n-2)|\nabla\ln(f')^{\frac{1}{2}}|^2\right)g_{jk}.
\end{aligned}$$

Now, by Gauss equation $R_{jk} = a_{jk} - b_{jk}$, the Ricci tensor R_{jk} is uniformly bounded because the tensors a_{jk} and b_{jk} are uniformly bounded. Indeed we have already prove that the tensor a_{jk} is uniformly bounded. To see that the tensor b_{jk} is uniformly bounded we know that $trace(b_{ij}) = |A|^2$ and since the tensor b_{ij} is positive defined, then $|b_{ij}|^2 \leq n|A|^2$.

Now, since ∇H, H are uniformly bounded, then from (4.39), the tensor $\tilde{R}_{jk}$ is uniformly bounded. Since the metrics $\tilde{g}_{jk}$ are uniformly equivalent, the last assumption, i.e (4.37) is also satisfied. So, we can apply the Harnack inequality theorem of Cao (theorem 4.3).

We are now ready to prove our main result (theorem 4.2):

Proof of the main theorem. We have already shown in Lemma 4.19 the first part of the theorem. As for the second part, we already know that the full norm of the second fundamental form and all its covariant derivatives are uniformly bounded so we have the longtime existence of the flow, i.e $T = \infty$. To get the convergence, we will use Harnack inequality theorem above and we will apply it to the function r. r is a positive function which satisfies

$$\frac{\partial}{\partial t} r = \tilde{\triangle} r.$$

Therefore it follows from the maximum principle for the parabolic equation that $\sup_{x\in M} r(x,t)$ is decreasing and $\inf_{x\in M} r(x,t)$ is increasing. As in Cao's paper ([5]) we now define for any integer $m > 1$

$$\phi_m(x,t) = \sup_{x\in M} r(x, m-1) - r(x, m-1+t)$$

$$\psi_m(x,t) = r(x, m-1+t) - \inf_{x\in M} r(x, m-1)$$

$$osc(t) = \sup_{x\in M} r(x,t) - \inf_{x\in M} r(x,t)$$

Since $\sup_{x\in M} r(x,t)$ is decreasing, then we have

$$\sup_{x\in M} r(x, m-1) > \sup_{x\in M} r(x, m-1+t).$$

Therefore ϕ_m is a positive function. All the same, since $\inf_{x\in M} r(x,t)$ is increasing, then we also have

$$\inf_{x\in M} r(x, m-1) < \inf_{x\in M} r(x, m-1+t).$$

We also conclude that ψ_m is a positive function. It is easy to check that ϕ_m and ψ_m hold the heat equation. Now, we apply the Harnack inequality theorem of Cao, above to these functions with $t_1 = \frac{1}{2}$ and $t_2 = 1$, and we obtain :

$$\sup_{x\in M} r(x, m-1) - \inf_{x\in M} r(x, m - \frac{1}{2}) \leq \gamma(\sup_{x\in M} r(x, m-1) - \sup_{x\in M} r(x, m))$$

$$\sup_{x\in M} r(x, m - \frac{1}{2}) - \inf_{x\in M} r(x, m-1) \le \gamma(\inf_{x\in M} r(x,m) - \inf_{x\in M} r(x, m-1))$$

where $\gamma > 1$ is a constant.
Add these last two inequalities, we have

$$osc(m-1) + osc(m - \frac{1}{2}) \le \gamma(osc(m-1) - osc(m)).$$

This implies that

$$osc(m-1) \le \gamma(osc(m-1) - osc(m))$$

and hence

$$osc(m) \le \delta osc(m-1),$$

with $\delta = \frac{\gamma-1}{\gamma} < 1$.
By induction we obtain

$$osc(m) \le \delta^m osc(0) \tag{4.40}$$

and $osc(0) = \sup_{x\in M} r(x,0) - \inf_{x\in M} r(x,0)$.
Since the oscillation function $osc(t)$ is decreasing in t, then we conclude from (4.40) that

$$osc(t) \le C_4 e^{-at} \tag{4.41}$$

where C_4 and a are positive constants that do not depend on t and $e^{-a} = \delta$. Therefore $osc(t)$ goes to zero as t goes to ∞. That implies that $r(x,t) = r \circ \alpha(x,t)$ converges uniformly to a constant as t goes to ∞. From (4.41), since by the maximum principle, $r = r(\alpha)$ is bounded, we can find a constant C_5 such that

$$|r(\alpha) - C_5| \le 2C_4 e^{-at}. \tag{4.42}$$

Now, since (L,g) is a compact Riemannian manifold, there exists a constant depending only on L and g such that for any smooth functions s the interpolation inequality

$$|\nabla s|^2 \le C_6|s|(|\nabla^2 s| + |\nabla s|)$$

holds. Since g_t is a family of uniformly equivalent metrics on L, then one can choose C_6 independent of t. From this we deduce that for $s = r(\alpha) - C_5$

$$|r'H|^2 \le C_6|r(\alpha) - C_5|(|r'\nabla H| + |r''H| + |r'H|).$$

Since $|\nabla H|^2 \le c(n)|\nabla A|^2 \le$constant, we get from (4.42)

$$|H|^2 \le C_7 e^{-at}$$

where C_7 is a constant independent of t. Consequently the mean curvature vector tends to zero exponentially. Now we can integrate the evolution equation $\frac{\partial}{\partial t}F = f'\overrightarrow{H}$ and the exponential decay of $|H|$ shows that for any $\epsilon > 0$ we can find a time t_0 such that for all $t \ge t_0$ the immersion L_t will stay in an ϵ-neighborhood of L_{t_0}. Then this proves convergence. The compactness of L implies that the limit manifold must be flat. q.e.d.

4.5 Selfsimilar solutions

A solution of (4.1) is called self-similar, if

$$\nabla_j f = c\langle F, \nu_j \rangle$$

for some constant c. A solution of this equation just moves by homotheties.

Theorem 4.4 *There are no closed Lagrangian selfsimilar solution of the flow (4.1).*

Proof: We have $H = d\alpha$, where α is a globally defined Lagrangian angle. Assume that $L \subset \mathbb{R}^{2n}$ is selfsimilar solution of the flow (4.1). That means that there exist a constant c such that $J\nabla f = cF^{\perp}$, which is equivalent to

$$\nabla_j f = c < F, \nu_j > . \tag{4.43}$$

Differentiating (4.43) once and using Lagrangian condition, we get

$$\nabla_i \nabla_j f = -c h_{ij}^k < F, F_k > . \tag{4.44}$$

Now if we take the trace of (4.44) we get

$$\triangle f + c < F, \nabla\alpha >= 0. \tag{4.45}$$

But $\triangle f := \nabla^i \nabla_i f = \nabla^i (f' H_i) = f''|H^2| + f' d^{\dagger} H$. Now $H = d\alpha$, so $\triangle f = f''|\nabla\alpha|^2 + f'\triangle\alpha$. So (4.45) becomes

$$f'\triangle\alpha + f''|\nabla\alpha|^2 + c < F, \nabla\alpha >= 0$$

which is an elliptic equation. Then by the strong maximum principle, we obtain that α is constant and then $H = 0$. This is a contradiction since there are no closed minimal submanifolds in euclidean space (see theorem 3.3).

q.e.d.

4.6 Monotonicity formula

In this section we ask, if there is a monotonicity formula for our flow.

To this end let $\rho(x,t)$ be the backward heat kernel at point $(0,t_0)$, i.e,

$$\rho(x,t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(-\frac{|x|^2}{4(t_0 - t)}\right) \qquad t < t_0.$$

Setting $\tau := t_0 - t$, we have :

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho(F(p,t),t) &= \frac{n}{2}4\pi\frac{1}{(4\pi\tau)^{n/2+1}}\exp\left(-\frac{|F(p,t)|^2}{4\tau}\right)\\
&\quad -\frac{1}{(4\pi\tau)^{n/2}}\left(\frac{2f'\langle F,H\rangle 4\tau+4|F|^2}{16\tau^2}\right)\exp\left(-\frac{|F(p,t)|^2}{4\tau}\right)\\
&= \frac{1}{(4\pi\tau)^{n/2}}\left(\frac{2n\pi}{4\pi\tau}-\frac{f'\langle F,H\rangle}{2\tau}-\frac{|F|^2}{4\tau^2}\right)\exp\left(-\frac{|F(p,t)|^2}{4\tau}\right)\\
&= \rho(F(p,t),t)\left(\frac{n}{2\tau}-\frac{f'\langle F,H\rangle}{2\tau}-\frac{|F|^2}{4\tau^2}\right).
\end{aligned}
$$

From $\frac{\partial}{\partial t}d\mu_t = -f'|H|^2 d\mu_t$ we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t}\rho(F(p,t),t)d\mu_t &= -\int_{M_t}\rho\left(f'|H|^2-\frac{n}{2\tau}+f'\frac{1}{2\tau}\langle F,H\rangle+\frac{|F|^2}{4\tau^2}\right)d\mu_t\\
&= -\int_{M_t}\rho\left(f'|H|^2-\frac{n}{2\tau}+f'\frac{1}{2\tau}\left\langle F^{\perp},H\right\rangle+\frac{|F|^2}{4\tau^2}\right)d\mu_t
\end{aligned}
$$

For $Y\in F^*(\mathbb{R}^{2n})$ we set $divY := g^{ij}\langle\nabla_{\partial_i}Y,F_j\rangle$. We have

$$
\begin{aligned}
divY &= g^{ij}\left\langle\nabla_{\partial_i}Y^{\top},F_j\right\rangle+g^{ij}\left\langle\nabla_{\partial_i}Y^{\perp},F_j\right\rangle\\
&= g^{ij}\left\langle\nabla^{\top}_{\partial_i}Y^{\top},F_j\right\rangle-g^{ij}\left\langle Y^{\perp},\nabla_{\partial_i}F_j\right\rangle\\
&= divY^{\top}-g^{ij}\left\langle Y,(\nabla_{\partial_i}F_j)^{\perp}\right\rangle\\
&= divY^{\top}-\langle H,Y\rangle,
\end{aligned}
$$

where $Y^{\top}$ is the tangent part of Y and $Y^{\perp}$ is its normal part. Since L is closed, then from divergence theorem, we obtain

$$
\int_{M_t} divY d\mu_t = -\int_{M_t}\langle H,Y\rangle\ d\mu_t. \tag{4.46}
$$

Now taking $Y=\frac{1}{2\tau}\rho F$, we have

$$
\begin{aligned}
divY &= \frac{1}{2\tau}\rho divF+\left\langle\nabla(\frac{1}{2\tau}\rho),F\right\rangle\\
&= \frac{n\rho}{2\tau}-g^{ij}\frac{\rho}{4\tau^2}\langle F,F_i\rangle\langle F,F_j\rangle\\
&= \frac{n\rho}{2\tau}-\frac{\rho}{4\tau^2}|F^{\top}|^2. \qquad (4.47)
\end{aligned}
$$

Then combining (4.46) and (4.47) we get :

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t}\rho(F(p,t),t)\;d\mu_t &= -\int_{M_t}\rho\left(f'|H|^2+\frac{(f'+1)}{2\tau}\left\langle F^{\perp},H\right\rangle+\frac{|F^{\perp}|^2}{4\tau^2}\right)d\mu_t\\
&= -\int_{M_t}\rho\left|\sqrt{f'}H+\frac{(f'+1)}{4\tau\sqrt{f'}}F^{\perp}\right|^2 d\mu_t\\
&\quad+\int_{M_t}\rho\left(\frac{(f'+1)^2}{16f'\tau^2}-\frac{1}{4\tau^2}\right)|F^{\perp}|^2\;d\mu_t.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{M_t}\rho(F(p,t),t)\;d\mu_t = &-\int_{M_t}\rho\left|\sqrt{f'}H+\frac{(f'+1)}{4\tau\sqrt{f'}}F^{\perp}\right|^2 d\mu_t\\
&+\frac{1}{16\tau^2}\int_{M_t}\rho\frac{(f'-1)^2}{f'}|F^{\perp}|^2\;d\mu_t.
\end{aligned}
$$

Unfortunately, the second term in the right hand side of this equation has a bad sign (positive one). So we don't get a monotonicity formula for this flow.

References

1. U. Abresch and J. Langer. The normalized curve shortening flow and homothetic solutions. *J. Differential Geom.*, 23(2):175–196, 1986.
2. Alessandro Arsie. Maslov class and minimality in Calabi-Yau manifolds. *J. Geom. Phys.*, 35(2-3):145–156, 2000.
3. Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.
4. Hubert Bray, Sean Hayward, Marc Mars, and Walter Simon. Generalized inverse mean curvature flows in spacetime. *Comm. Math. Phys.*, 272(1):119–138, 2007.
5. Huai Dong Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81(2):359–372, 1985.
6. Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.
7. J. Y. Chen; J. Y. Li: Singularities of Codimension two Mean Curvature Flow of Symplectic Surfaces, math.DG/0208227 (2002).
8. Jing Yi Chen, Jia Yu Li, and Gang Tian. Two-dimensional graphs moving by mean curvature flow. *Acta Math. Sin. (Engl. Ser.)*, 18(2):209–224, 2002.
9. Kai Cieliebak and Edward Goldstein. A note on the mean curvature, Maslov class and symplectic area of Lagrangian immersions. *J. Symplectic Geom.*, 2(2):261–266, 2004.
10. Klaus Ecker. Estimates for evolutionary surfaces of prescribed mean curvature. *Math. Z.*, 180(2):179–192, 1982.
11. Klaus Ecker and Gerhard Huisken. Mean curvature evolution of entire graphs. *Ann. of Math. (2)*, 130(3):453–471, 1989.
12. Klaus Ecker and Gerhard Huisken. A Bernstein result for minimal graphs of controlled growth. *J. Differential Geom.*, 31(2):397–400, 1990.
13. S. D. Eidelman. *Parabolic systems.* Translated from the Russian by Scripta Technica, London. North-Holland Publishing Co., Amsterdam, 1969.
14. L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.
15. Atsushi Fujioka. Surfaces with harmonic inverse mean curvature in space forms. *Proc. Amer. Math. Soc.*, 127(10):3021–3025, 1999.
16. Atsushi Fujioka and Jun-ichi Inoguchi. Spacelike surfaces with harmonic inverse mean curvature. *J. Math. Sci. Univ. Tokyo*, 7(4):657–698, 2000.

17. Claus Gerhardt. Evolutionary surfaces of prescribed mean curvature. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. I (Paris, 1978/1979)*, volume 53 of *Res. Notes in Math.*, pages 193–250. Pitman, Boston, Mass., 1981.
18. Claus Gerhardt. The inverse mean curvature flow in Robertson-Walker spaces and its application to cosmology. *Methods Appl. Anal.*, 13(1):19–28, 2006.
19. Claus Gerhardt. The inverse mean curvature flow in cosmological spacetimes. *Adv. Theor. Math. Phys.*, 12(6):1183–1207, 2008.
20. K. Groh, M. Schwarz, K. Smoczyk, and K. Zehmisch. Mean curvature flow of monotone Lagrangian submanifolds. *Math. Z.*, 257(2):295–327, 2007.
21. Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.
22. Bernhard Hein. A homotopy approach to solving the inverse mean curvature flow. *Calc. Var. Partial Differential Equations*, 28(2):249–273, 2007.
23. Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.
24. Gerhard Huisken and Tom Ilmanen. Higher regularity of the inverse mean curvature flow. *J. Differential Geom.*, 80(3):433–451, 2008.
25. Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.
26. Gerhard Huisken. Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. *Invent. Math.*, 84(3):463–480, 1986.
27. Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.
28. Gerhard Huisken. Local and global behaviour of hypersurfaces moving by mean curvature. *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math.*, 54:175–191, 1993
29. Yoshinobu Kamishima. Geometric flows on compact manifolds and global rigidity. *Topology*, 35(2):439–450, 1996.
30. N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(1):75–108, 1983.
31. N. V. Krylov. *Nonlinear elliptic and parabolic equations of the second order*, volume 7 of *Mathematics and its Applications (Soviet Series)*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky.
32. Jean-Marie Morvan. Classe de Maslov d'une immersion lagrangienne et minimalité. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(13):633–636, 1981.
33. Roger Moser. The inverse mean curvature flow and p-harmonic functions. *J. Eur. Math. Soc. (JEMS)*, 9(1):77–83, 2007.
34. Roger Moser. The inverse mean curvature flow as an obstacle problem. *Indiana Univ. Math. J.*, 57(5):2235–2256, 2008.
35. André Neves. Singularities of Lagrangian mean curvature flow: zero-Maslov class case. *Invent. Math.*, 168(3):449–484, 2007.
36. Isabel M. C. Salavessa. On the Kähler angles of submanifolds. *Port. Math. (N.S.)*, 60(2):215–235, 2003.
37. R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134(3-4):275–288, 1975.
38. Felix Schulze. Evolution of convex hypersurfaces by powers of the mean curvature. *Math. Z.*, 251(4):721–733, 2005.
39. Felix Schulze. Convexity estimates for flows by powers of the mean curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 5(2):261–277, 2006.

40. K. Smoczyk; G. Wang; Y. L. Xin: Mean curvature flow with flat normal bundles. math.DG/0411010v1 (2004).
41. Knut Smoczyk. Nonexistence of minimal Lagrangian spheres in hyperKähler manifolds. *Calc. Var. Partial Differential Equations*, 10(1):41–48, 2000.
42. Knut Smoczyk. Angle theorems for the Lagrangian mean curvature flow. *Math. Z.*, 240(4):849–883, 2002.
43. Knut Smoczyk. Longtime existence of the Lagrangian mean curvature flow. *Calc. Var. Partial Differential Equations*, 20(1):25–46, 2004.
44. K. Smoczyk: Der Lagrangesche mittlere Krümmungsfluss. (The Lagrangian mean curvature flow). (German) Leipzig: Univ. Leipzig (Habil.-Schr.), 102 pages, (2000).
45. Knut Smoczyk and Mu-Tao Wang. Mean curvature flows of Lagrangians submanifolds with convex potentials. *J. Differential Geom.*, 62(2):243–257, 2002.
46. Knut Smoczyk. Self-shrinkers of the mean curvature flow in arbitrary codimension. *Int. Math. Res. Not.*, (48):2983–3004, 2005.
47. Knut Smoczyk. Remarks on the inverse mean curvature flow. *Asian J. Math.*, 4(2):331–335, 2000.
48. Jian Song and Ben Weinkove. On Donaldson's flow of surfaces in a hyperkähler four-manifold. *Math. Z.*, 256(4):769–787, 2007.
49. Jeffrey D. Streets. Quasi-local mass functionals and generalized inverse mean curvature flow. *Comm. Anal. Geom.*, 16(3):495–537, 2008.
50. Michael Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2000. Applications to nonlinear partial differential equations and Hamiltonian systems.
51. R. Temam. Applications de l'analyse convexe au calcul des variations. In *Nonlinear operators and the calculus of variations (Summer School, Univ. Libre Bruxelles, Brussels, 1975)*, pages 208–237. Lecture Notes in Math., Vol. 543. Springer, Berlin, 1976.
52. Mu-Tao Wang. Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension. *Invent. Math.*, 148(3):525–543, 2002.
53. Mu-Tao Wang. Gauss maps of the mean curvature flow. *Math. Res. Lett.*, 10(2-3):287–299, 2003.
54. Mao-Pei Tsui and Mu-Tao Wang. A Bernstein type result for special Lagrangian submanifolds. *Math. Res. Lett.*, 9(4):529–535, 2002.
55. Brian White. The size of the singular set in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.*, 13(3):665–695 (electronic), 2000.